Exact solutions for the two- and all-terminal reliabilities of the Brecht-Colbourn ladder and the generalized fan

Christian Tanguy

France Telecom Division R&D, CORE/MCN/OTT, 38–40 rue du Général Leclerc, 92794 Issy-les-Moulineaux Cedex, France

Abstract

The two- and all-terminal reliabilities of the Brecht-Colbourn ladder and the generalized fan have been calculated exactly for arbitrary size as well as arbitrary individual edge and node reliabilities, using transfer matrices of dimension four at most. While the all-terminal reliabilities of these graphs are identical, the special case of identical edge (p) and node (ρ) reliabilities shows that their two-terminal reliabilities are quite distinct, as demonstrated by their generating functions and the locations of the zeros of the reliability polynomials, which undergo structural transitions at $\rho = \dfrac{1}{2}$.

Key words: network reliability, transfer matrix, zeros of the reliability polynomial, algebraic structures

1 Introduction

Network reliability has long been a practical issue since the pioneering work of Moore and Shannon [46], and will remain addressed by reliability engineers, statistical physicists and applied mathematicians for years, since networks are pervading our everyday life. Not surprisingly, the study of network reliability has led to a huge body of literature, which includes excellent textbooks and surveys [6,7,26,59,60]. In the following, we consider a probabilistic approach of reliability, in which the network is represented by an undirected graph $G = (V, E)$, where V is a set of nodes (also called vertices), and E is a set of undirected edges (or links). Each element of V and E has a probability

---

Email address: `christian.tanguy@orange-ft.com` (Christian Tanguy).

p_n or p_e to operate correctly; failures of constituents are assumed to occur at random, and to be statistically independent events (this restriction may be relaxed, eventually). Among the different measures of reliability, one often considers the k-terminal reliability, i.e., the probability that a given subset K of k nodes ($K \subset E$) are connected. The most common instances are the all-terminal reliability Rel_A ($K \equiv E$) and the two-terminal reliability $\text{Rel}_2(s \to t)$, which deals with a particular connection between a source s and a terminal destination t. Both of them are affine functions of *each* p_n and p_e.

The sheer number of possible system states, namely $2^{|E|+|V|}$, clearly precludes the use of an "enumeration of states" strategy for realistic networks, and shows that the final expression may be extremely cumbersome. Consequently, most studies have assumed graphs with perfect nodes ($p_n \equiv 1$) and edges of identical reliability p; all reliabilities are then expressed as a polynomial in p, called the reliability polynomial. Radio broadcast networks [35] have also been described by networks with perfectly reliable edges but imperfect nodes; in this context, one speaks of residual connectedness [12]. It was shown early on [26] that the calculation of k-terminal reliability is #P-hard, even after the simplifying and restricting assumptions that (i) the graph is planar (ii) all nodes are perfectly reliable (iii) all edges have the same reliability p.

The difficulty of the problem has stimulated many approaches : partitioning techniques [30,77], sum of disjoint products [2,5,38,53,63], graph simplifications (series-parallel reductions [46], triangle-star (also called delta-wye) transformations [20,31,34,55,72], factoring [42]), determination of various lower and upper bounds to reliability polynomials [6,13,14,17,22,26,27,52,58], Monte-Carlo simulations [33,40,47], genetic [23] and ordered binary decision diagram (OBDD) algorithms [43,54,75,76]. The reliability polynomial has also been studied [51] with the aim of deriving some useful and hopefully general information from the structure of its coefficients [21,28], or the location of its zeros in the complex plane [16].

The Tutte polynomial $T(G, x, y)$ of a graph G has also been shown to be equivalent to the Potts model partition function of the q-state Potts model [61,73]. Calculations for various recursive families of graphs G quickly followed [19]. The all-terminal reliability polynomial $\text{Rel}_A(G, p)$ of graphs is deduced from $T(G, 1, \frac{1}{1-p})$. Royle and Sokal [56] proved that the Brown-Colbourn conjecture [16] on the location of the zeros of $\text{Rel}_A(G, p)$, while valid for series-parallel reducible graphs, does not hold for a few families of graphs. While these results are extremely valuable to better understand a few properties of graphs and all-terminal reliability polynomials, they still assume that nodes are perfect and that edges have the same reliability.

In recent years, the growth of Internet traffic has called for a better evaluation of the reliability of connections in — among others, optical — networks. This,

of course, strongly depends on the connection under consideration. Actual failure rates and maintenance data show that a proper evaluation of two-terminal reliabilities must put node and edge equipments on an equal footing, i.e., both edge (fiber links, optical amplifiers) and node (optical cross-connects, routers) failures must be taken into account. The possibility of node failure has been considered in early papers [1,32,36], to quote but a few. Adaptation of algorithms to include imperfect nodes has been addressed, sometimes controversially [48,41,70,71,76]; the two-variable approach for bounds to the reliability polynomial, by Bulka and Dugan [18] and Chen and He [22], is also worth mentioning. In order to be realistic, different edge reliabilities should be used: for instance, the failure rate of optical fiber links is likely to increase with their length.
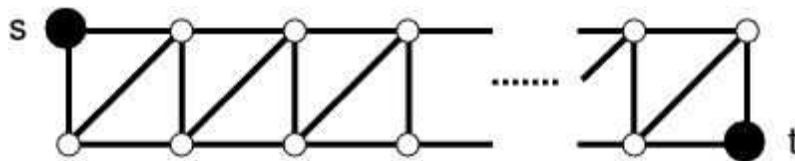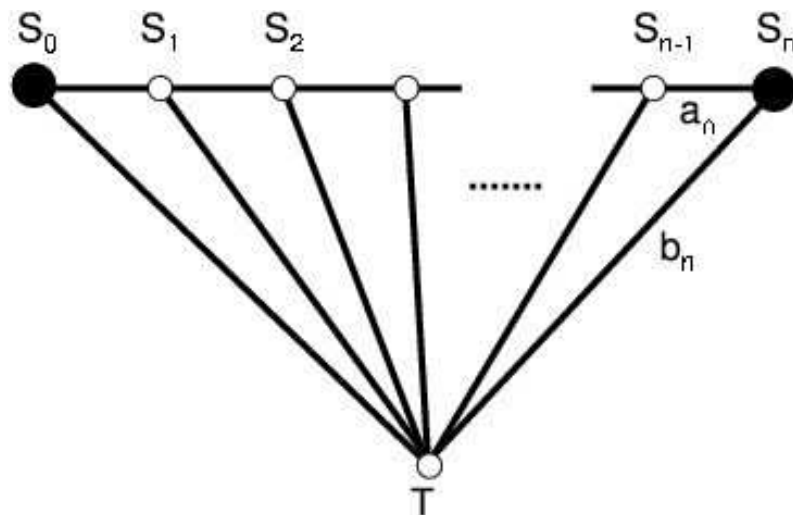

Fig. 1. The Brecht-Colbourn ladder.

Fig. 2. The generalized fan.

In this work, we give the exact two-terminal reliability of two well-known graphs, namely the Brecht-Colbourn ladder [13,14,52] and the generalized fan [3,49], which have the same all-terminal reliability. Both networks have been studied in the literature, especially the first one, as a case study for different lower bounds of the two-terminal reliability [14], but with a limited number of

3

nodes. Here, the number of nodes will be arbitrary. Our solution fully exploits the recursive nature of the graphs and the triangle-star transformation with *unreliable* nodes [34]. The final expressions are products of (at most) 4×4 transfer matrices, in which the arbitrary reliability of each node and edge appears explicitly. In the case of identical edge (p) and node (ρ) reliabilities, we provide the generating functions of the two-terminal reliabilities, leading to a very simple expression for the generalized fan. Another byproduct is the asymptotic locations of the zeros of the two-terminal reliability polynomials for the Brecht-Colbourn ladder and the generalized fan, which exhibit very different structures and undergo a structural transition at $\rho = \dfrac{1}{2}$.

Our aim is (i) to give a detailed derivation of the final results, so that people involved in reliability studies can readily use an easy-to-implement formula (ii) compare the exact solution with previous results (iii) exhibit the structural changes undergone by the location of the zeros of reliability polynomials as a function of node reliability (iv) emphasize anew the importance of algebraic structures of the underlying graphs in the determination of their associated polynomials [10,59].

Our paper is organized as follows : In Section 2, we briefly recall the formulae for the triangle-star transformation for unreliable nodes. In Section 3, we define the notations for the different edge and node reliabilities and detail the derivation of the main results (eqs. (12) and (14) for the Brecht-Colbourn ladder, eqs. (23) and (25) for the generalized fan). Section 4 is devoted to the case where all edges and nodes have identical reliabilities p and ρ, respectively, the size of the network appearing simply as an integer n. We give the analytical solution of the two-terminal reliability $\mathrm{Rel}_2(p, \rho; n)$ for the Brecht-Colbourn ladder and the generalized fan, and the associated generating functions, which encode all the necessary information in a beautifully simple, compact form. The asymptotic power-law or constant behaviors are given when $n \to \infty$. Prompted by the nearly universal character of the Brown-Colbourn conjecture [16], we address in Section 5 the location of zeros of the two-terminal reliability polynomials, and show that their structures are very different even though their internal structure is similar. For the sake of completeness, we derive in Section 6 the common all-terminal reliability for both networks, for arbitrary values of edge reliabilities. Finally, we conclude by indicating several directions in which the present results may be further extended, so that, for instance, a catalog of exactly solvable networks — in terms of reliability — may be given rapidly [69]. Such a catalog of elementary bricks could be useful for a new and improved set of bounds or benchmarks for alternative methods in the general case.

2 Triangle-star transformation for unreliable nodes

The triangle-star — also called delta-wye or $\Delta - Y$ — transformation has been used many times to simplify calculations of network reliability [20,26,31,34,42,55,72]. It has mostly been applied in a perfect nodes context, to provide upper and lower bounds to the exact reliability. Here, we exploit this transformation to the full in the case of imperfect nodes in order to obtain exact results. Since it plays a crucial part of the derivation, we give the formulae derived by Gadani [34].
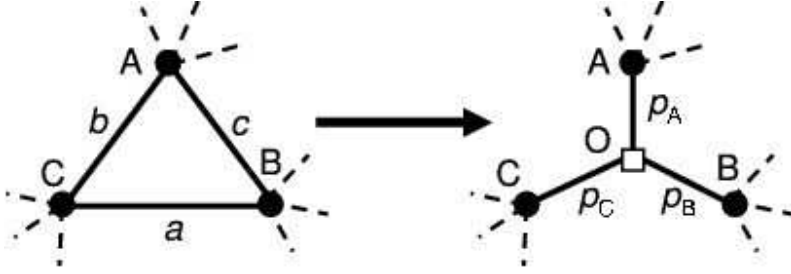

Fig. 3. Triangle-star transformation for unreliable nodes. A, B, and C are the node reliabilities, with a, b, and c the edge reliabilities of the initial network, and p_A, p_B, p_C, and O, those of the transformed network.

Let us consider three particular nodes A, B, and C of a network, connected as shown in the left part of Fig. 3 (the reliability of the nodes are given by the same uppercase variables, so as to prevent an unnecessary multiplication of notations). The reliability of the edge connecting A and B is given by (lowercase) c, with similar notation for the remaining edges of the triangle. The aim of the triangle-star transformation is to replace the triangle by a star, which is possible by the addition of a new *unreliable* node O and three new edges connecting O to A, B, and C, with reliabilities p_A, p_B, and p_C, respectively. Both networks are equivalent provided that the following compatibility relations hold [34]

$$p_A\,O\,p_C = b + a\,c\,B - a\,b\,c\,B, \tag{1}$$
$$p_A\,O\,p_B = c + a\,b\,C - a\,b\,c\,C, \tag{2}$$
$$p_B\,O\,p_C = a + b\,c\,A - a\,b\,c\,A, \tag{3}$$
$$p_A\,O\,p_B\,p_C = a\,b + b\,c + a\,c - 2\,a\,b\,c. \tag{4}$$

Note that the first three equalities correspond to the probability that the two nodes under consideration are connected, while the last one gives the probability that the three nodes are connected. A word of caution — already

5

given by Gadani — is worth mentioning in the case of *successive* triangle-star transformation: the triangles should have no common edge or node.

3   Derivation of the main results

3.1   Brecht-Colbourn ladder

Let us first name the different edge and node reliabilities of the Brecht-Colbourn ladder, and detail how we can use the triangle-star transformation of the preceding section. Assuming that S_0 is always the source node, and using node reliabilities S_i, we note a_n the reliability of the edge (S_{n-2}, S_n), and b_n that of (S_{n-1}, S_n), as shown at the top of Fig. 4.

The application of the triangle-star transformation to Fig. 4 leads to

$$p_1\, p_0\, p_2 = b_n + a_n\, b_{n-1}\, S_{n-2} - a_n\, b_{n-1}\, b_n\, S_{n-2}, \tag{5}$$
$$p_1\, p_0\, p_3 = a_n + b_{n-1}\, b_n\, S_{n-1} - a_n\, b_{n-1}\, b_n\, S_{n-1}, \tag{6}$$
$$p_1\, p_0\, p_2\, p_3 = a_n\, b_n + a_n\, b_{n-1} + b_{n-1}\, b_n - 2\, a_n\, b_{n-1}\, b_n. \tag{7}$$

Writing the two-terminal reliability $\mathrm{Rel}_2^{(\mathrm{BC})}(S_0 \to S_n) = S_n\, \mathcal{S}_n$, we immediately see that

$$\mathcal{S}_n = p_1\, p_0\, \mathcal{S}_{n-1}(a_{n-1} \to a_{n-1}\, S_{n-1}\, p_2,\ b_{n-1} \to p_3), \tag{8}$$

because the transformed graph remains essentially a Brecht-Colbourn ladder, provided that some edges are renormalized (b_{n-1} must be replaced by p_3, S_{n-1} by p_0, and a_{n-1} by $a_{n-1}\, S_{n-1}\, p_2$).

In the case of imperfect nodes, the assumption

$$\mathcal{S}_n = \alpha_n\, a_n + \widetilde{\alpha}_n\, a_n\, S_{n-1} + \widetilde{\beta}_n\, b_n\, S_{n-1} + \widetilde{\gamma}_n\, a_n\, b_n\, S_{n-1} \tag{9}$$

gives four parameters, for which we want to find a recursion relation. When we use eqs. (5)–(7) in eq. (9), there is one slightly tricky point: we are actually dealing with Boolean functions, so that $S_{n-2}^2 \equiv S_{n-2}$ (see for instance [3]) and

$$p_1\, p_0\, \widetilde{\alpha}_{n-1}\, S_{n-2}\, (a_{n-1}\, S_{n-1}\, p_2)$$
$$= \widetilde{\alpha}_{n-1}\, S_{n-1}\, S_{n-2}\, a_{n-1}\, (b_n + a_n\, b_{n-1}\, S_{n-2} - a_n\, b_{n-1}\, b_n\, S_{n-2})$$
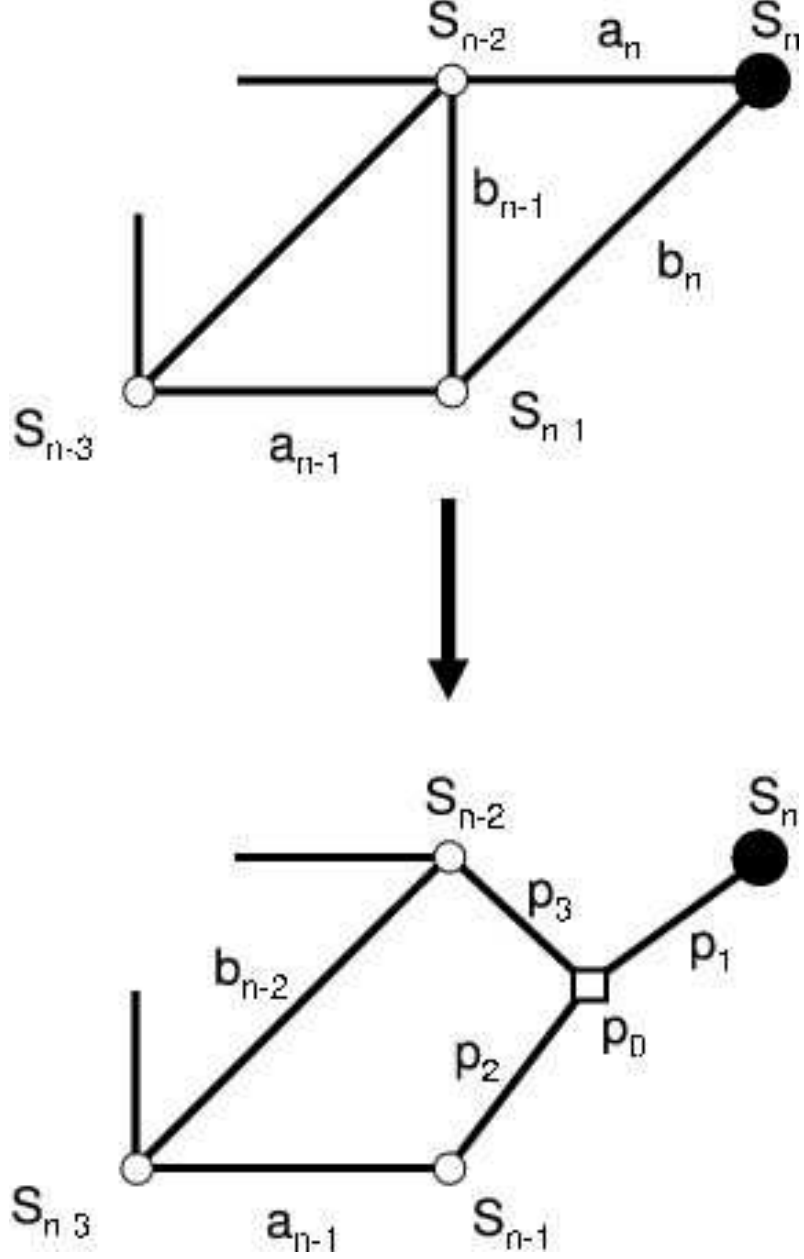
6

Fig. 4. Decomposition of the ladder using the triangle-star transformation.

$$\equiv \widetilde{\alpha}_{n-1}\, S_{n-1}\, S_{n-2}\, a_{n-1}\, (b_n + a_n\, b_{n-1} - a_n\, b_{n-1}\, b_n). \tag{10}$$

We deduce

$$\begin{pmatrix} \alpha_n \\ \widetilde{\alpha}_n \\ \widetilde{\beta}_n \\ \widetilde{\gamma}_n \end{pmatrix} = M_{n-1} \cdot \begin{pmatrix} \alpha_{n-1} \\ \widetilde{\alpha}_{n-1} \\ \widetilde{\beta}_{n-1} \\ \widetilde{\gamma}_{n-1} \end{pmatrix}, \tag{11}$$

where the 4×4 transfer matrix M_{n-1} is given by

$$M_{n-1} = \begin{pmatrix} 0 & 0 & S_{n-2} & 0 \\ a_{n-1}\, b_{n-1}\, S_{n-2} & a_{n-1}\, b_{n-1}\, S_{n-2} & 0 & a_{n-1}\, b_{n-1}\, S_{n-2} \\ a_{n-1} & a_{n-1}\, S_{n-2} & b_{n-1}\, S_{n-2} & a_{n-1}\, b_{n-1}\, S_{n-2} \\ -a_{n-1}\, b_{n-1}\, S_{n-2} & -a_{n-1}\, b_{n-1}\, S_{n-2} & -b_{n-1}\, S_{n-2} & a_{n-1}\,(1 - 2\, b_{n-1})\, S_{n-2} \end{pmatrix} . \quad (12)$$

For $n = 2$, the two-terminal reliability is easily solved because we have series-parallel graph:

$$\mathrm{Rel}_2^{(\mathrm{BC})}(S_0 \to S_2) = S_2\,(a_2 + b_1\, S_1\, b_2 - a_2\, b_1\, S_1\, b_2)\, S_0, \quad (13)$$

which leads to $\alpha_2 = S_0$, $\widetilde{\alpha}_2 = 0$, $\widetilde{\beta}_2 = b_1\, S_0$, and $\widetilde{\gamma}_2 = -b_1\, S_0$, or equivalently to $\alpha_1 = \widetilde{\alpha}_1 = \widetilde{\gamma}_1 = 0$ et $\widetilde{\beta}_1 = 1$ (see eq. (12)). The final result is

$$\mathrm{Rel}_2^{(\mathrm{BC})}(S_0 \to S_n) = S_n\, (0\ 0\ 1\ 0) \cdot M_n \cdot M_{n-1}\ \cdots\ M_1 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \quad (14)$$

Note that in M_1, a_1 is quite arbitrary, so that it can be set equal to zero without loss of generality. Equation (14) also holds for n equal to 1 or even 0 (following a frequent convention that a product of zero matrices is the identity matrix); it has also been independently checked for the first values of n, using a sum of disjoint products procedure.

In the case of perfect nodes, the ansatz should be $\mathcal{S}_n = \alpha_n\, a_n + \beta_n\, b_n + \gamma_n\, a_n\, b_n$. The relevant 3×3 transfer matrix M'_n is now

$$M'_n = \begin{pmatrix} a_n\, b_n & 1 & a_n\, b_n \\ a_n & b_n & a_n\, b_n \\ -a_n\, b_n & -b_n & a_n\,(1 - 2\, b_n) \end{pmatrix}, \quad (15)$$

from which we deduce

$$\mathrm{Rel}_2^{(\mathrm{BC})}(S_0 \to S_n; \text{perfect nodes}) = (0\ 1\ 0) \cdot M'_n \cdot M'_{n-1}\ \cdots\ M'_1 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \quad (16)$$
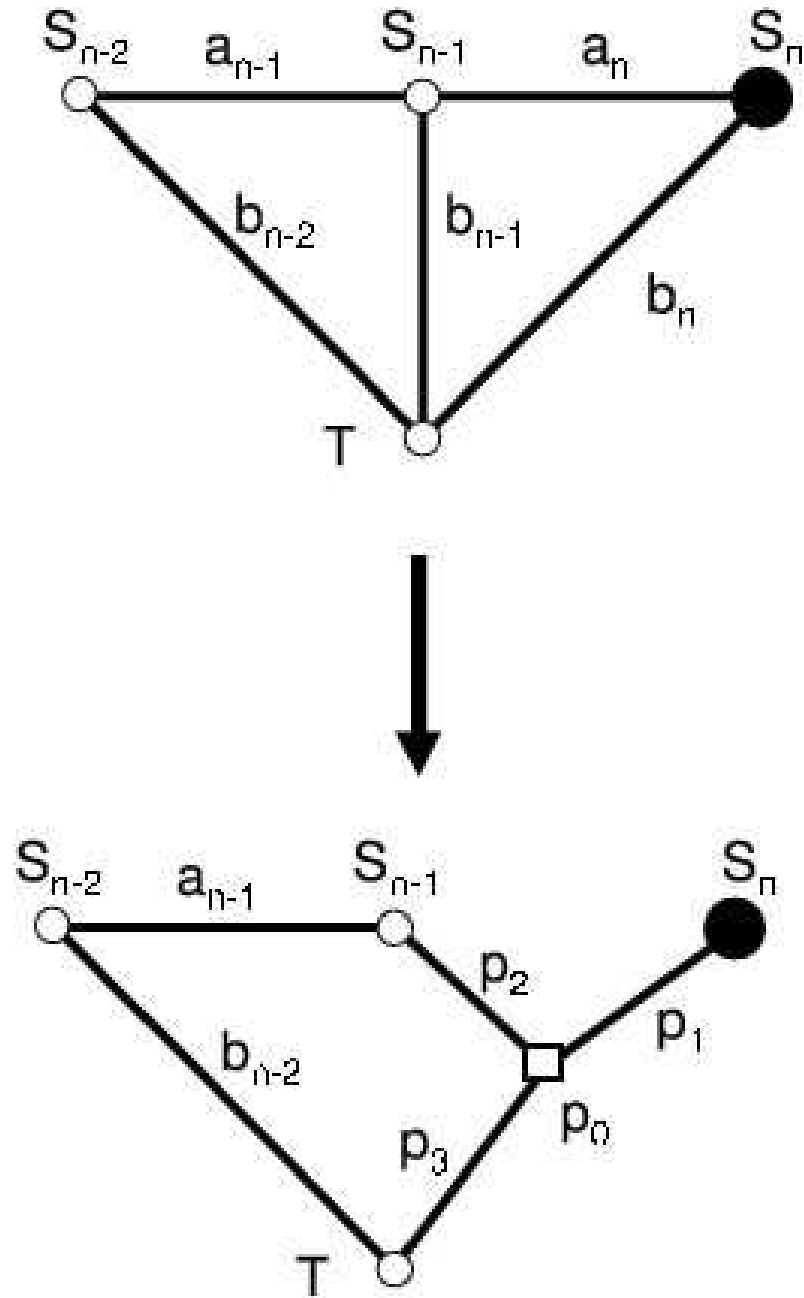
Fig. 5. Triangle-star transformation for the generalized fan.

Let us now turn to the second architecture. The triangle-star transformation, applied to the generalized fan, is displayed on Fig. 5, and we expect the common node T to play a special role in the recursion relation. We can write

$\mathrm{Rel}_2^{(\mathrm{fan})}(S_0 \to S_n) = S_n \, \widehat{\mathcal{S}}_n$, and see immediately that

$$\widehat{\mathcal{S}}_n = p_1 \, p_0 \, \widehat{\mathcal{S}}_{n-1}(a_{n-1} \to a_{n-1} \, S_{n-1} \, p_2, b_{n-1} \to p_3), \tag{17}$$

with

$$p_1 \, p_0 \, p_2 = a_n + b_{n-1} \, b_n \, T - a_n \, b_{n-1} \, b_n \, T, \tag{18}$$
$$p_1 \, p_0 \, p_3 = b_n + a_n \, b_{n-1} \, S_{n-1} - a_n \, b_{n-1} \, b_n \, S_{n-1}, \tag{19}$$
$$p_1 \, p_0 \, p_2 \, p_3 = a_n \, b_n + a_n \, b_{n-1} + b_{n-1} \, b_n - 2 \, a_n \, b_{n-1} \, b_n. \tag{20}$$

Our ansatz is now

$$\widehat{\mathcal{S}}_n = S_n \, \alpha_n \, a_n + \widetilde{\alpha}_n \, a_n \, T + \widetilde{\beta}_n \, b_n \, T + \widetilde{\gamma}_n \, a_n \, b_n \, T. \tag{21}$$

Here again, we must be careful with the Boolean variable T. When using eq. (18) in eq. (21), all occurrences of T^2 must be replaced by T. The recursion relations take the now familiar form

$$\begin{pmatrix} \alpha_n \\ \widetilde{\alpha}_n \\ \widetilde{\beta}_n \\ \widetilde{\gamma}_n \end{pmatrix} = \widehat{M}_{n-1} \cdot \begin{pmatrix} \alpha_{n-1} \\ \widetilde{\alpha}_{n-1} \\ \widetilde{\beta}_{n-1} \\ \widetilde{\gamma}_{n-1} \end{pmatrix}, \tag{22}$$

where the transfer matrix $\widehat{M}_n$ is

$$\widehat{M}_n = \left(\begin{array}{c|c|c|c} a_n \, S_n & 0 & 0 & 0 \\ \hline 0 & a_n \, S_n & b_n \, S_n & a_n \, b_n \, S_n \\ \hline a_n \, b_n \, S_n & a_n \, b_n \, S_n & 1 & a_n \, b_n \, S_n \\ \hline -a_n \, b_n \, S_n & -a_n \, b_n \, S_n & -b_n \, S_n & a_n \, (1 - 2 \, b_n) \, S_n \end{array} \right). \tag{23}$$

When $n = 1$, we have a series-parallel graph:

$$\mathrm{Rel}_2^{(\mathrm{fan})}(S_0 \to S_1) = S_1(a_1 + b_0 \, b_1 \, T - a_1 \, b_0 \, b_1 \, T) \, S_0, \tag{24}$$

from which we deduce the final expression

$$\text{Rel}_2^{(\text{fan})}(S_0 \to S_n) = (1 \; T \; 0 \; 0) \cdot \widehat{M}_n \cdot \widehat{M}_{n-1} \cdots \widehat{M}_1 \cdot \widehat{M}_0 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad (25)$$

with the additional convention $a_0 \equiv 1$ for $\widehat{M}_0$. We have also checked the correctness of eq. (25) by a sum of disjoint products procedure for the first values of n. It also agrees with the expression of Aggarwal et $al.$ [3] for $n = 3$.

In the case of perfect nodes, which is often considered in the literature, the ansatz $\widehat{\mathcal{S}}_n = \alpha_n \, a_n + \beta_n \, b_n + \gamma_n \, a_n \, b_n$ leads to

$$\text{Rel}_2^{(\text{fan})}(S_0 \to S_n; \text{perfect nodes}) = (1 \; 0 \; 0) \cdot \widehat{M'}_n \cdot \widehat{M'}_{n-1} \cdots \widehat{M'}_1 \cdot \widehat{M'}_0 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, (26)$$

with

$$\widehat{M'}_n = \begin{pmatrix} \begin{array}{c|c|c} a_n & b_n & a_n \, b_n \\ \hline a_n \, b_n & 1 & a_n \, b_n \\ \hline -a_n \, b_n & -b_n & a_n \, (1 - 2\, b_n) \end{array} \end{pmatrix}, \qquad (27)$$

and again $a_0 \equiv 1$ for $\widehat{M}_0$.

It is worth noting that while the sizes of M_n and $\widehat{M}_n$ for the Brecht-Colbourn and fan cases are identical, their matrix elements are clearly distinct. This actually leads to strong differences, as will be shown in the following section.

4 Identical reliabilities p and ρ

4.1 Introduction

While eqs. (12) and (14) give the two-terminal reliability for the general Brecht-Colbourn ladder, and eqs. (23) and (25) for the fan, we consider here

the special case where all edges and nodes have reliabilities p and ρ, respectively. Because identical edge reliabilities are usually taken for granted, our exact results may help to demonstrate once again the underlying connection between combinatorics and reliability theory, most particularly in the enumeration of self-avoiding walks [26,35,52]. This assumption greatly simplifies the problem, since all transfer matrices, with the exception of $\widehat{M}_0$ in the case of the generalized fan, are identical to a matrix $M(p, \rho)$ (or $\widehat{M}(p, \rho)$): the two-terminal reliability is essentially the nth power of one matrix. While the determination of its eigenvalues and eigenvectors is one way to solve the problem, we adopt in the following the generating function formalism, which is a fundamental tool in combinatorics [65], and makes for a very compact synthesis of the results. The generating function $G(z)$ is defined by

$$G(z) = \sum_{n=0}^{\infty} \mathrm{Rel}_2^{(n)}(p, \rho)\, z^n. \tag{28}$$

From the finite-order recursion relations obeyed by $\mathrm{Rel}_2^{(n)}$, the generating function is obviously a rational function of z, namely $G(z) = N(z)/D(z)$. In the present study, its denominator is at most of degree 4 in z, because in the absence of further simplification, $D(z) = z^4\, P_{\text{carac}}(1/z)$, where P_{carac} is the characteristic polynomial of the transfer matrix. The determination of $N(z)$ is then very straightforward: we only need to multiply the first terms of the expansion of $G(z)$ by $D(z)$ to see the numerator $N(z)$ emerge. The partial fraction decomposition of $G(z)$ finally leads to a compact, sometimes very simple, analytical expression of $\mathrm{Rel}_2^{(n)}(p, \rho)$, for arbitrary n.

4.2 Brecht-Colbourn ladder

4.2.1 Imperfect nodes and edges

When edge and node reliabilities are p et ρ, the transfer matrix of eq. (12) is equal to

$$M(p, \rho) = \left(\begin{array}{c|c|c|c} 0 & 0 & \rho & 0 \\ \hline p^2\,\rho & p^2\,\rho & 0 & p^2\,\rho \\ \hline p & p\,\rho & p\,\rho & p^2\,\rho \\ \hline -p^2\,\rho & -p^2\,\rho & -p\,\rho & p\,(1-2\,p)\,\rho \end{array}\right), \tag{29}$$

and

$$\mathrm{Rel}_2(S_0 \to S_n) = \rho\,(0\ 0\ 1\ 0) \cdot M(p, \rho)^n \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$ (30)

The characteristic polynomial of this matrix is

$$P_{\mathrm{carac}}^{(\mathrm{BC})}(p, \rho; x) = x^4 - p\,(2 - p)\,\rho\,x^3 - p\,\rho\,(1 - \rho\,(p + p^2 - p^3))\,x^2$$
$$+ (1 - p)\,(1 - p\,\rho)\,p^2\,\rho^2\,x - (1 - p)\,(1 - \rho)\,p^4\,\rho^3.$$ (31)

The eigenvalues of $M(p, \rho)$ are given by the roots of this polynomial of degree 4. From the first few values of $\mathrm{Rel}_2(S_0 \to S_n)$ (taking $n = 2, ..., 10$, for instance), we can easily deduce the generating function $G_{\mathrm{BC}}^{(0)}(z)$.

$$G_{\mathrm{BC}}^{(0)}(z) = p\,\rho^2\,z^2\,\frac{N_{\mathrm{BC}}^{(0)}(z)}{D_{\mathrm{BC}}(z)},$$ (32)

with

$$N_{\mathrm{BC}}^{(0)}(z) = 1 + p\,(1 - p)\,\rho + p^2\,\rho(1 - p\,\rho\,(2 - p))\,z - p^2\,\rho^2\,(1 - p)^2\,z^2$$
$$+ p^4\,(1 - p)\,\rho^3\,(1 - \rho)\,z^3$$ (33)
$$D_{\mathrm{BC}}(z) = 1 - p\,(2 - p)\,\rho\,z - p\,\rho\,(1 - \rho\,(p + p^2 - p^3))\,z^2$$
$$+ (1 - p)\,(1 - p\,\rho)\,p^2\,\rho^2\,z^3 - (1 - p)\,(1 - \rho)\,p^4\,\rho^3\,z^4$$ (34)

Adding $\rho + p\,\rho^2\,z$ (basically, the two-terminal reliability of a single node, or that of two connected nodes) to $G_{\mathrm{BC}}^{(0)}(z)$, the numerator further simplifies, so that we may use another generating function, which of course provides the same coefficient of z^n for $n \geq 2$, namely

$$G_{\mathrm{BC}}(z) = \rho\,\frac{1 - p\,(1 - p)\,\rho\,z + p^3\,(1 - p)\,\rho^2\,z^2}{D_{\mathrm{BC}}(z)};$$ (35)

$N_{\mathrm{BC}}(z)$ will be the numerator of $G_{\mathrm{BC}}(z)$ in eq. (35).

What is the nature of the roots ? Numerically, two situations occur when $0 \leq p \leq 1$ and $0 \leq \rho \leq 1$: there are either (i) four real roots or (ii) two real roots and two complex roots. For instance, when $\rho = 0.9$, there are four real roots if $p < 0.5533938...$, but only two otherwise. The separation of the

two domains occurs when one root is degenerate. When this happens, the equalities $P_{\text{carac}}^{(\text{BC})}(p, \rho; x) = 0$ and $\dfrac{\partial P_{\text{carac}}^{(\text{BC})}(p, \rho; x)}{\partial x} = 0$ lead to a polynomial constraint satisfied by p and ρ, given in the appendix (eq. (A.5)). For a given ρ, there exists a unique solution in the range $0 < p < 1$, $p_{\text{crit}}(\rho)$, which is displayed in Fig. 6.
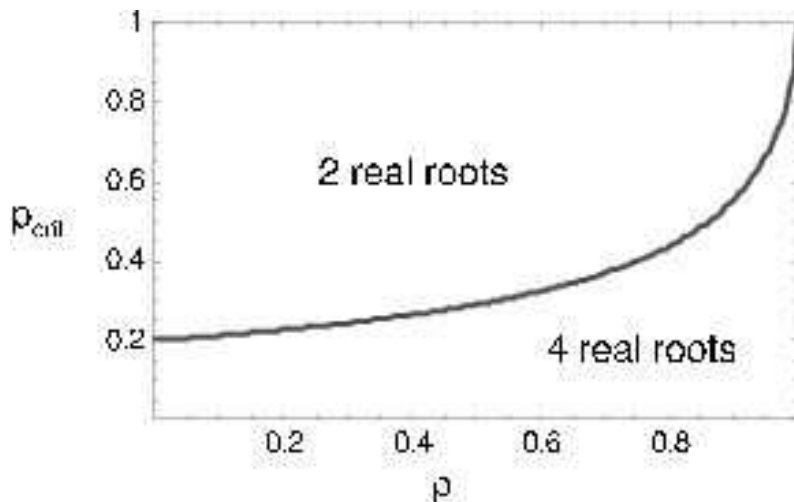

Fig. 6. Separation of the domains (ρ, p) for which four real roots exist (under the curve) or two real and two complex roots coexist (above it).

From the inspection of eq. (35), we may wonder when numerator and denominator of the generating function may share a common root. A Gröbner basis calculation shows that this happens only when $p = 2$, outside our range of interest. Therefore, there are four distinct roots most of the times, and when there is a double (real) root of $\mathcal{P}(p_{\text{crit}}(\rho), \rho; x)$, it is indeed degenerate (there is no simplification of $G_{\text{BC}}(z)$). The exact expression for $\text{Rel}_2(S_0 \to S_n)$ can then be deduced using a partial fraction decomposition of $G_{\text{BC}}(z)$. When all the eigenvalues λ_i are distinct,

$$G_{\text{BC}}(z) = \sum_{\lambda_i} \underbrace{\frac{-\lambda_i \, N_{\text{BC}}(\frac{1}{\lambda_i})}{D'_{\text{BC}}(\frac{1}{\lambda_i})}}_{\alpha_i} \frac{1}{1 - \lambda_i z}; \tag{36}$$

consequently, the two-terminal reliability reads

$$\text{Rel}_2(S_0 \to S_n) = \sum_{\lambda_i} \alpha_i \, \lambda_i^n. \tag{37}$$

In the Brecht-Colbourn case, the root $\lambda_{\max}$ of greatest modulus is always real. Unless p and ρ are equal to 1, $\lambda_{\max} < 1$, so that $\text{Rel}_2^{(\text{BC})}(S_0 \to S_n)$ decreases

14

as $n \to \infty$, essentially as a power-law behavior

$$\text{Rel}_2^{(\text{BC})}(S_0 \to S_n) \sim \lambda_{\max}^n. \tag{38}$$

4.2.2 Imperfect edges and perfect nodes

We devote this section to the case of imperfect edges and perfect nodes, since the Brecht-Colbourn ladder was initially investigated in this configuration [13,14]. We immediately find

$$\text{Rel}_2^{(\text{BC};\rho=1)}(S_0 \to S_n) = (0 \ 1 \ 0) \cdot \begin{pmatrix} \begin{array}{c|c|c} p^2 & 1 & p^2 \\ \hline p & p & p^2 \\ \hline -p^2 & -p & p\,(1-2\,p) \end{array} \end{pmatrix}^n \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \tag{39}$$

so that the 25-node case [14] leads to

$$\begin{aligned}
\text{Rel}_2(S_0 \to S_{24}) = p^{12}\,(1 &+ 78\,p + 1121\,p^2 + 6633\,p^3 + 11554\,p^4 - 57525\,p^5 \\
&- 279450\,p^6 + 89578\,p^7 + 2570040\,p^8 + 1431183\,p^9 \\
&- 17159566\,p^{10} - 12166498\,p^{11} + 98985590\,p^{12} \\
&+ 33119917\,p^{13} - 495566666\,p^{14} + 212008622\,p^{15} \\
&+ 1867178285\,p^{16} - 2888906214\,p^{17} - 3066846055\,p^{18} \\
&+ 14427083319\,p^{19} - 14178781875\,p^{20} - 17955754991\,p^{21} \\
&+ 80808979717\,p^{22} - 144754404751\,p^{23} + 174732303288\,p^{24} \\
&- 158925117297\,p^{25} + 113702258108\,p^{26} - 65190712312\,p^{27} \\
&+ 30133254848\,p^{28} - 11197537798\,p^{29} + 3308571601\,p^{30} \\
&- 761406139\,p^{31} + 131805146\,p^{32} - 16170009\,p^{33} \\
&+ 1254886\,p^{34} - 46368\,p^{35}).
\end{aligned} \tag{40}$$

The coefficients are already quite large, even though the number of nodes remains limited. To the best of our knowledge, the exact two-terminal reliability polynomial was calculated only for a 10-node ladder [13]; we recover this result by taking $n = 9$ in eq. (39).

The three distinct real eigenvalues λ_i $(i = 1, 2, 3)$ of the transfer matrix appearing in eq. (39), one of which is negative, are displayed as a function of p in Fig. 7.

Their analytical expression is ($\xi \in \{1, e^{2i\pi/3}, e^{-2i\pi/3}\}$)

$$\lambda = \frac{1}{3}\,p\,(2-p) + \frac{1}{3}\left[\xi\left(\frac{\mathcal{A}+i\sqrt{27}\,\sqrt{\mathcal{B}}}{2}\right)^{1/3} + \xi^*\left(\frac{\mathcal{A}-i\sqrt{27}\,\sqrt{\mathcal{B}}}{2}\right)^{1/3}\right], \tag{41}$$
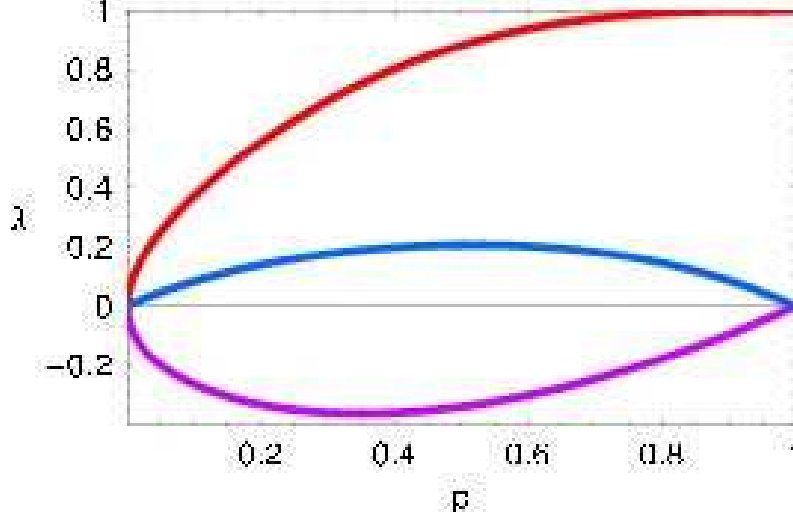
15

Fig. 7. Variation with p of the three eigenvalues for the Brecht-Colbourn ladder with perfect nodes (see eq. (41)).

where

$$\mathcal{A} = -p^2 \left(9 - 43\,p + 60\,p^2 - 39\,p^3 + 11\,p^4\right), \tag{42}$$

$$\mathcal{B} = (1-p)^2\,p^3 \left(4 + 9\,p + 16\,p^2 - 88\,p^3 + 98\,p^4 - 32\,p^5 - 8\,p^6 + 5\,p^7\right); \tag{43}$$

note that $\mathcal{B}$ is always positive for $0 < p < 1$.

The generating function for the Brecht-Colbourn ladder with perfect nodes is simply

$$G_{\mathrm{BC}}^{(\rho=1)}(z) = \frac{1 - p\,(1-p)\,z + p^3\,(1-p)\,z^2}{1 - p\,(2-p)\,z - p\,(1-p)^2\,(1+p)\,z^2 + p^2\,(1-p)^2\,z^3}. \tag{44}$$

From its partial fraction decomposition, which is necessarily of the form $G_{\mathrm{BC}}^{(\rho=1)}(z) = \sum_{i=1}^{3} \dfrac{\alpha_i}{1 - \lambda_i\,z}$ because the eigenvalues are distinct, we deduce after simplification

$$\mathrm{Rel}_2(S_0 \to S_n) = \sum_{i=1}^{3} \underbrace{\frac{p\,(1-p)^2 - (1-p)\,\lambda_i - \lambda_i^2}{3\,p\,(1-p)^2 - 2(1-p)^2\,(1+p)\,\lambda_i - (2-p)\,\lambda_i^2}}_{\alpha_i} \,\lambda_i^n. \tag{45}$$

From eq. (45), it is clear that the large n limit of $\mathrm{Rel}_2(S_0 \to S_n)$ is a power-law behavior, in which the eigenvalue of maximum modulus quickly prevails over the others when p is close to unity, even for moderate values of n.

16

Table 1
Values of the F_i's for the two-terminal reliability polynomial of the 25-node Brecht-Colbourn ladder.

i	F_i	i	F_i	i	F_i	i	F_i
0	1	9	1125395882	18	762855455898	27	3042073238
1	47	10	3974128827	19	800820887863	28	635100751
2	1079	11	12199394435	20	725278875430	29	105465538
3	16103	12	32708854487	21	562806091836	30	13648753
4	175418	13	76833130394	22	371300292894	31	1334810
5	1484837	14	158368734141	23	206539411448	32	93929
6	10151340	15	286502593795	24	96061397122	33	4368
7	57524387	16	454444238576	25	37052347922	34	113
8	275139029	17	630595957484	26	11756780232	35	1

We can also write $\mathrm{Rel}_2(S_0 \rightarrow S_n) = \sum_{i=0}^{2\,n-1} F_i\, p^{2\,n-1-i}\,(1-p)^i$, a well-known expansion of reliability polynomials [26]. The F_i's for the 25-node ladder are given in Table 1 (the maximum value was equal to 8078 in the 10-node ladder [13]). Let us call $m = n+1$ the number of nodes. Using linear regressions on the first values of $\mathrm{Rel}_2(S_0 \rightarrow S_n)$, it is straightforward to find

$$
\begin{aligned}
\mathrm{Rel}_2^{(m\ \mathrm{odd})}(p) =\ & p^{\frac{m-1}{2}}\,(1-p)^{\frac{3\,m-5}{2}} + \frac{m^2 - 12\,m - 21}{8}\,p^{\frac{m+1}{2}}\,(1-p)^{\frac{3\,m-7}{2}} \\
& + \frac{m^4 + 72\,m^3 + 350\,m^2 - 2376\,m + 2337}{384}\,p^{\frac{m+3}{2}}\,(1-p)^{\frac{3\,m-9}{2}} \\
& + \cdots
\end{aligned}
\tag{46}
$$

$$
\begin{aligned}
\mathrm{Rel}_2^{(m\ \mathrm{even})}(p) =\ & \frac{m}{2}\,p^{\frac{m}{2}}\,(1-p)^{\frac{3\,m-6}{2}} + \frac{(m-2)\,(m^2 + 38\,m + 24)}{48}\,p^{\frac{m+2}{2}}\,(1-p)^{\frac{3\,m-8}{2}} \\
& + \frac{(m-2)\,(m^4 + 122\,m^3 + 2304\,m^2 - 5472\,m - 13440)}{3840}\,p^{\frac{m+4}{2}}\,(1-p)^{\frac{3\,m-10}{2}} \\
& + \cdots
\end{aligned}
\tag{47}
$$

Finally, the comparison of the exact results with the various lower bounds proposed in [14] is given in Table 2 and Fig. 8. It shows that the lower bound

Table 2
Comparison of various lower bounds with the exact two-terminal reliability of a 25-node Brecht-Colbourn ladder [14].

p	Kruskal-Katona	MinCost(edp)	Brecht-Colbourn	exact
0.75	0.031682	0.054681	0.054681	0.625163
0.80	0.068803	0.119917	0.430912	0.773696
0.82	0.092654	0.161200	0.558991	0.824038
0.84	0.124041	0.214282	0.669269	0.867950
0.86	0.165305	0.281396	0.761945	0.905042
0.88	0.219694	0.364529	0.837486	0.935251
0.90	0.291856	0.464826	0.896659	0.958806
0.91	0.336579	0.521297	0.920440	0.968231
0.92	0.388392	0.581555	0.940574	0.976194
0.93	0.448415	0.644934	0.957259	0.982785
0.94	0.517724	0.710375	0.970720	0.988109
0.95	0.597041	0.776313	0.981207	0.992275
0.96	0.686113	0.840514	0.989003	0.995400
0.97	0.782518	0.899895	0.994420	0.997608
0.98	0.879474	0.950274	0.997801	0.999024
0.99	0.961964	0.986085	0.999524	0.999778

of Brecht and Colbourn is rather good for p close to unity, but its sharpness decreases for $p < 0.8$.

4.2.3 Imperfect nodes and perfect edges

The configuration of imperfect nodes and perfect edges, although less studied than the previous one, has nonetheless been considered in several papers [1,12,35]. The Brecht-Colbourn ladder case has been studied in detail by Graver and Sobel [35]. The relevant transfer matrix is now equal to

$$M(p=1,\rho) = \begin{pmatrix} 0 & 0 & \rho & 0 \\ \rho & \rho & 0 & \rho \\ 1 & \rho & \rho & \rho \\ -\rho & -\rho & -\rho & -\rho \end{pmatrix}. \tag{48}$$
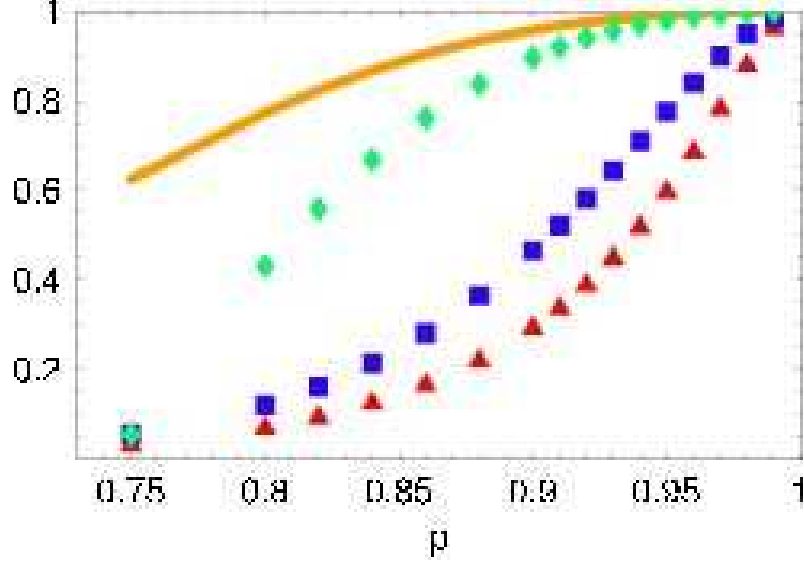
18

Fig. 8. Comparison of the exact value (———) with lower bounds given in ref. [14] for the 25-node ladder:(▲) Kruskal-Katona (■) Min Cost (♦) Brecht-Colbourn.

Its characteristic polynomial is $x^2 \left(x^2 - \rho\, x - \rho\,(1 - \rho)\right)$, which simply leads to the eigenvalues 0 — which plays no role in the final expression of the two-terminal reliability — and $\lambda_\pm = \dfrac{1}{2}\left(\rho \pm \sqrt{4\,\rho - 3\,\rho^2}\right)$. The generating function is found to be

$$G_{\mathrm{BC}}^{(p=1)}(z) = \frac{\rho}{1 - \rho\, z - \rho\,(1 - \rho)\, z^2} = \frac{\rho}{\lambda_+ - \lambda_-}\left(\frac{\lambda_+}{1 - \lambda_+\, z} - \frac{\lambda_-}{1 - \lambda_-\, z}\right), \quad (49)$$

which gives

$$\mathrm{Rel}_2(S_0 \to S_n) = \frac{\rho}{\sqrt{4\,\rho - 3\,\rho^2}}\left(\lambda_+^{n+1} - \lambda_-^{n+1}\right). \quad (50)$$

This is almost Graver and Sobel's result [35], obtained through a combinatorial argument. Their final expression differs from ours because their source and destination (in our notation, S_0 and S_n) are perfect.

19

4.3 Generalized fan

When the reliabilities are p et ρ, the transfer matrix of eq. (23) is equal to

$$\widehat{M}(p,\rho) = \left(\begin{array}{c|c|c|c} p\,\rho & 0 & 0 & 0 \\ \hline 0 & p\,\rho & p\,\rho & p^2\,\rho \\ \hline p^2\,\rho & p^2\,\rho & 1 & p^2\,\rho \\ \hline -p^2\,\rho & -p^2\,\rho & -p\,\rho & p\,(1-2\,p)\,\rho \end{array}\right) \tag{51}$$

and

$$\mathrm{Rel}_2(S_0 \to S_n) = \frac{1}{p}\,(1\ \rho\ 0\ 0)\cdot\widehat{M}(p,\rho)^{n+1}\cdot\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \tag{52}$$

the $1/p$ prefactor being a consequence of the condition $a_0 \equiv 1$ in $\widehat{M}_0$. The characteristic polynomial of this matrix factorizes nicely:

$$P_{\mathrm{carac}}^{(\mathrm{fan})}(p,\rho;x) = (x-1)\,(x-p\,\rho)\,\Big(x - p\,(1-p)\,\rho\Big)^2, \tag{53}$$

so that the eigenvalues are 1, $p\,\rho$ and $p\,(1-p)\,\rho$, the latter being of degree 2. The presence of 1 among the roots should not be surprising. Indeed, even when n goes to infinity, the two-terminal reliability between S_0 and S_n is larger than $p\,\rho^2$, in stark contrast to the Brecht-Colbourn case, where the reliability vanishes. Since a power-law behavior is still expected, the only possibility is that the largest eigenvalue is of modulus 1.

The generating function $G^{(\mathrm{fan})}(z)$ is derived using the recipe described in the Brecht-Colbourn case. The additional convention $\mathrm{Rel}_2(S_0 \to S_0) = \rho$ leads to

$$G^{(\mathrm{fan})}(z) = \frac{N^{(\mathrm{fan})}(z)}{D^{(\mathrm{fan})}(z)}, \tag{54}$$

$$N^{(\mathrm{fan})}(z) = 1 - z\,[1 + p\,\rho\,(1-p)\,(2-p\,\rho)] + p\,\rho\,z^2\,[(2+p\rho)\,(1-p)$$
$$+ p^2\,\rho\,(p-\rho)] - p^2\,(1-p)^2\,\rho^2\,z^3, \tag{55}$$

$$D^{(\mathrm{fan})}(z) = (1-z)\,(1-p\,\rho\,z)\,(1-p\,(1-p)\,\rho\,z)^2. \tag{56}$$

Because all the roots have simple expressions, we expect to find a simple, analytical expression for $\mathrm{Rel}_2^{(\mathrm{fan})}(S_0 \to S_n)$. The partial fraction decomposition of eq. (54) gives indeed

$$G^{(\mathrm{fan})}(z) = \frac{\rho^2 (1-p)}{(1-p(1-p)\rho)^2} \frac{1 - 2p(1-p)\rho + p^2 \rho^2 (1-3p+p^2)}{1-p(1-p)\rho z}$$

$$+ p\rho^2 \frac{1 - p\rho(2-p)}{1-p(1-p)\rho} \frac{1}{(1-p(1-p)\rho z)^2}$$

$$+ \frac{p^2 \rho^3}{(1-p(1-p)\rho)^2} \frac{1}{1-z} + \rho(1-\rho) \frac{1}{1-p\rho z}. \tag{57}$$

The new feature of eq. (57) is the $(1 - p(1-p)\rho z)^{-2}$ term. Because

$$\frac{1}{(1-\lambda z)^2} = \sum_{n=0}^{\infty} (n+1) \lambda^n z^n, \tag{58}$$

the final expression of the two-terminal reliability is

$$\mathrm{Rel}_2^{(\mathrm{fan})}(S_0 \to S_n) = +p^n (1-p)^n \rho^{n+2} \left(n\, p \frac{1-p\rho(2-p)}{1-p(1-p)\rho} \right.$$

$$\left. + \frac{1 - p\rho(2-p) + p^2(1-p)^2 \rho^2}{(1-p(1-p)\rho)^2} \right)$$

$$+ p^n \rho^{n+1} (1-\rho) + \frac{p^2 \rho^3}{(1-p(1-p)\rho)^2}, \tag{59}$$

where n appears in a prefactor, not only as an exponent. The second term of eq. (59) is the asymptotic limit when $n \to \infty$. We see that the reliability of the path $S_0 \to T \to S_n$ is enhanced by $1/(1-p(1-p)\rho)^2$. When nodes are perfect, ρ must be set to one in eq. (59), and the contribution of the eigenvalue $p\rho$ vanishes (the transfer matrix of eq. (27) is 3×3). This leads to

$$\mathrm{Rel}_2^{(\mathrm{fan};\rho=1)}(S_0 \to S_n) = p^n (1-p)^{n+2} \left(\frac{n\, p}{1-p(1-p)} + \frac{(1+p^2)}{(1-p(1-p))^2} \right)$$

$$+ \frac{p^2}{(1-p(1-p))^2}. \tag{60}$$

21

5 Zeros of the two-terminal reliability polynomials

The structure of the different reliability polynomials may be understood by studying the locations of their zeros in the complex plane. Such a study has been fruitfully performed in the case of the chromatic polynomial [9,11,57], most notably in the context of the four-color theorem. In reliability studies, some effort has been done to discover general properties for the all-terminal reliability $\mathrm{Rel}_A(p)$ [21,28,51], its main byproduct being the Brown-Colbourn conjecture [16], according to which all the zeros are to be found in the region $|1 - p| < 1$. Although valid for series-parallel graphs, this remarkable conjecture does not strictly hold in the general case (but not by far) [56]. As mentioned in the introduction, the all-reliability polynomial is linked to the Tutte polynomial, an invariant of the graph. It has also been studied extensively by Chang and Shrock for various recursive families of graphs [19], who give the limiting curves where all zeros of the polynomials converge.

In this section, after briefly recalling general results of the literature, we study the roots of $\mathrm{Rel}_2(p, \rho) = 0$ in the complex plane for a fixed ρ, in order to see whether some insight may also be gained in this case. Admittedly, this polynomial depends on the couple (source, terminal), but structures are still expected. We show that the zeros tend to aggregate along portions of algebraic curves, which can substantially differ even for the two families of graphs under consideration, even though they have the same all-terminal reliability. Moreover, structural transitions occur at $\rho = \frac{1}{2}$.

5.1 Calculation of the limiting curves (generalities)

As n grows, the number of zeros of the reliability polynomial in the complex plane increases. Because of the matrix transfer property, we have recursion relations between relability polynomials corresponding to successive values of n. The general treatment of the problem has been done by Beraha, Kahane, and Weiss [8], but may be understood in the following, simplifying way: if the reliability polynomial is of the form $\sum_i \alpha_i \lambda_i(p)^n$ (where λ_i are the eigenvalues of the transfer matrix), then at large n, only the two eigenvalues of greater modules, say λ_1 and λ_2, will prevail, so that the reliability polynomial will vanish when $|\lambda_1(p)| = |\lambda_2(p)|$ (of course, it might be three or more eigenvalues of equal modulus; the present oversimplification works quite well here). This defines a set of curves in the complex plane, where all zeros should accumulate in the $n \to \infty$ limit. The interested reader should refer to the work of Salas and Sokal [57] for a very detailed discussion of the convergence to the limiting curves. This behavior is not modified when one or more of the α_i's is a polynomial in n [8], as will become apparent for the generalized fan.
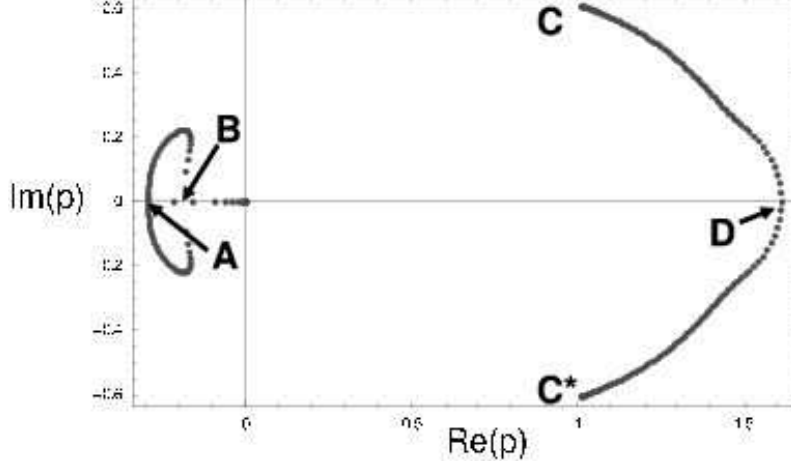
Fig. 9. Localization of the complex roots p of the two-terminal reliability polynomial for a Brecht-Colbourn ladder of 151 perfect nodes and 299 edges.

5.2   Calculation for the Brecht-Colbourn ladder

The recursion relation obeyed by $\mathrm{Rel}_2^{(\mathrm{BC})}(S_0 \to S_n) \equiv \mathrm{Rel}_2^{(n)}$ is easily deduced from eq. (31); it reads

$$\mathrm{Rel}_2^{(n)}(p) = p\,(2-p)\,\rho\,\mathrm{Rel}_2^{(n-1)} + p\,\rho\,(1 - p\,\rho - p^2\,\rho + p^3\,\rho)\,\mathrm{Rel}_2^{(n-2)}$$
$$-p^2\,(1-p)\,\rho^2\,(1-\rho)\,\mathrm{Rel}_2^{(n-3)} + p^4\,(1-p)\,\rho^3\,(1-\rho)\,\mathrm{Rel}_2^{(n-4)}. \quad (61)$$

and leads to very quick calculations using mathematical softwares such as Mathematica [45]. Note that this recursion relation of order four becomes of order three when $\rho = 1$. We can then look for solutions of $\mathrm{Rel}_2^{(n)}(p) = 0$ in the complex plane. Of course, the degree of the polynomial, as well as the magnitude of its coefficients, will increase with n.

5.2.1   Perfect nodes

Let us begin by setting $\rho = 1$. A first step is to display the roots of $\mathrm{Rel}_2^{(n)}(p) = 0$. These zeros have been calculated using the `NSolve[]` routine of Mathematica; because the polynomial coefficients can be very large, a numerical accuracy of several hundred digits is sometimes necessary. Figure 9 shows the location of the zeros of $\mathrm{Rel}_2^{(150)}(p)$ in the complex plane (it corresponds to a graph with 151 nodes and 299 edges).

On the right half of the complex plane, we have a simple, open curve crossing the real axis at D and extremities at C and C^*. On the left half, however, we have what looks like a closed curve — intersecting the negative real axis in A and B — as well as some dots on the negative real axis, between A and the

origin.

As mentioned above, it is well known that as n increases, the zeros accumulate at particular locations, constituted by segments/portions of algebraic curves. For $n = 150$, these limits are almost reached, even though the "sampling" is not uniform. We can calculate the limiting curves by considering that the roots of the characteristic polynomial are $\lambda\, e^{i\theta/2}$, $\lambda\, e^{-i\theta/2}$ and μ. The coefficients of the characteristic polynomial imply constraints between λ (which may be complex), μ and θ, or more precisely, $t = \cos\frac{\theta}{2}$ or $T = \cos\theta$, which will be used in the following. For instance, $\mu = (2 - p)\,p - 2\,\lambda\,\cos\frac{\theta}{2}$. After eliminating λ and μ, we obtain a compatibility condition which must be satisfied by p, and T (the polynomial $\mathcal{P}_3(p, \rho, T)$ is given in eq. (A.3)):

$$
\begin{aligned}
\mathcal{P}_3(p, \rho = 1, T) = {} & 2 + 9\,p - 4\,p^2 - 30\,p^3 + 38\,p^4 - 10\,p^5 - 6p^6 + 3\,p^7 \\
& + \left(2 + 12\,p - 8\,p^2 - 34\,p^3 + 48\,p^4 - 18\,p^5 - 2\,p^6 + 2\,p^7 \right) T \\
& - 4\,p\,(1 - p)^2 \left(1 - p + p^2 \right) T^2 - 8\,p\,(1 - p)^2\,T^3 \\
= {} & 0.
\end{aligned}
\tag{62}
$$

The additional requirement $|\mu| < |\lambda|$ translates into an additional contraint between p and T.

A plot of this parametric set of curves shows indeed that the whole portion of the negative real axis between A and the origin is indeed a solution, so that we could expect more zeros there when $n > 150$.

A few critical points can also be deduced from eq. (62). For instance, D corresponds to a real solution of $\mathcal{P}_3(p, \rho = 1, T = -1) = 0$, namely $p_D = \frac{1 + \sqrt{5}}{2}$. C and C^* are points of the complex plane defined by p_C and its complex conjugate, which are roots of $\mathcal{P}_3(p, \rho = 1, T = +1) = 0$, that is

$$
4 + 9\,p + 16\,p^2 - 88\,p^3 + 98\,p^4 - 32\,p^5 - 8\,p^6 + 5\,p^7 = 0.
\tag{63}
$$

This polynomial already appeared in the expression of $\mathcal{B}$. Note that there are more than one pair of complex solutions to this equation, the relevant one is given by $p_C \approx 1.011578 + i\,0.607394$. The determination of the complex numbers p_A and p_B associated with A and B is a little more elaborate. Both p_A and p_B are real and negative; they correspond to roots of degree two of eq. (62). In order to find them, we must have $\mathcal{P}_3(p, \rho = 1, T) = 0$ and $\dfrac{\partial \mathcal{P}_3(p, \rho = 1, T)}{\partial p} = 0$. These two relations are satisfied for special values of p (or T). After substitution and elimination of T using Mathematica, we obtain a product of polynomials, which must cancel for $p = p_A$ and $p = p_B$. For the sake of completeness, we provide the needed polynomials in appendix A. A

24

close numerical study allows to identify the relevant polynomials:

- $p_A \approx -0.2879878$ is solution of $\mathcal{P}_1(p, \rho = 1) = 0$ (see eq. (A.1)), that is

$$0 = 1 - p - 11\,p^2 + 15\,p^3 - 3\,p^4 - 2\,p^5 - p^6 + p^7, \tag{64}$$

attained for $T \approx 0.138176$, a root of

$$0 = -1 + 11\,T - 27\,T^2 - 26\,T^3 + 140\,T^4 + 240\,T^5 + 144\,T^6 + 32\,T^7. \tag{65}$$

- $p_B \approx -0.1849482$ is solution of $\mathcal{P}_2(p, \rho = 1) = 0$ (see eq. (A.2)), i.e.,

$$
\begin{aligned}
0 = {} & 1 - 4\,p - 9\,p^2 + 101\,p^3 - 413\,p^4 + 1019\,p^5 - 1761\,p^6 \\
& + 2151\,p^7 - 1864\,p^8 + 1097\,p^9 - 386\,p^{10} + 60\,p^{11},
\end{aligned} \tag{66}
$$

attained for $T \approx -0.9511957$, one of the solutions of

$$
\begin{aligned}
0 = {} & 3040707 + 12576464\,T + 15322821\,T^2 - 4376828\,T^3 \\
& - 22559186\,T^4 - 10112842\,T^5 + 9510320\,T^6 + 7762048\,T^7 \\
& - 1337920\,T^8 - 2068608\,T^9 + 8192\,T^{10} + 204800\,T^{11}
\end{aligned} \tag{67}
$$

5.2.2 Imperfect nodes

We can perform the study for $\rho < 1$ by following the same lines as above. Although more cumbersome because there are now four possible roots instead of three, the calculations are interesting nonetheless because we may vary one parameter, ρ. A natural question is: can it affect the global structure of the zeros ? The answer is yes, and the location of the zeros undergoes a kind of "structural transition" for a particular value of ρ, namely $\frac{1}{2}$.

Let us first consider $\rho > \frac{1}{2}$. The location of the zeros is then qualitatively similar to that described in Fig. 9, the whole structure merely expanding from the origin. After comparison with the numerical values for $n = 150$, D appears to be associated with $p_D = \frac{1}{2}(1 + \sqrt{1 + \frac{4}{\rho}})$, the relevant solution of $\mathcal{P}_3(p, \rho, T = -1) = 0$. Likewise, C and C^* are given by p_C, which is one complex solution of $\mathcal{P}_3(p, \rho, T = +1) = 0$ ($\mathcal{P}_3$ is given in eq. (A.3)).

The determination of p_A and p_B is more tedious, because $\mathcal{P}_3(p, \rho, T)$ is now a polynomial of degree 6 in T. Here again, we must find the common zeros of $\mathcal{P}_3(p, \rho, T)$ and its derivative with respect to p. The elimination of T, performed using Mathematica, leads to a polynomial in p and ρ, which must vanish. Actually, this polynomial can be factored, and a numerical comparison with the zeros obtained for $n = 150$ shows that p_A is solution of $\mathcal{P}_1(p, \rho) = 0$ (see eq. (A.1)). Similarly, p_B is a solution of $\mathcal{P}_2(p, \rho) = 0$, where $\mathcal{P}_2$ is given in eq. (A.2).
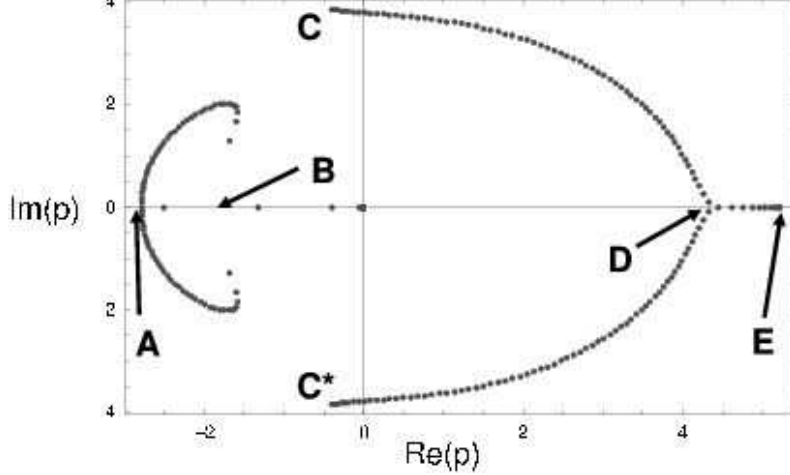
Fig. 10. Localization of the complex roots p of the two-terminal reliability polynomial for a Brecht-Colbourn ladder of 151 nodes and 299 edges, when $\rho = 10^{-2}$.

For $\rho = \frac{1}{2}$, $p_A \approx -0.4359355$, $p_B \approx -0.2885759$, $p_C \approx 0.748541 + i\,1.03759$, and $p_D = 2$. Note that p_A and p_B are zeros of polynomials in p of degrees equal to 22 and 30, respectively (having $\rho \neq 1$ does not simplify the problem).

For $\rho < \frac{1}{2}$, the above expressions for p_A, p_B, and p_C are still valid, even though C and C^* may now belong to the left half-plane. However, on the right half-plane, the structure of zeros undergoes a drastic transformation, as shown in Fig. 10 for $\rho = 10^{-2}$. D is now an angular point, and zeros may be found on the real axis between D and E. After numerical comparisons between structures of zeros and limiting curves have been performed, it appears that p_D is now a real root of $\mathcal{P}_1(p, \rho) = 0$ (actually, the third one, in decreasing order), while p_E is another real root of $\mathcal{P}_3(p, \rho, T = +1) = 0$.

5.2.3 Asymptotic limits when $\rho \to 0$

In this section, we address the asymptotic dependence as ρ decreases to zero of all the critical points A, B, C, D, and E. A first step consists in finding the corresponding p's for (very) small values of ρ. It is not too difficult to observe that these numerical values have a $\rho^{-1/3}$ dependence. We can then use improve on this knowledge by using the algebraic equations of which they are solutions, making again use of Mathematica for the asymptotic expansions. A comparison with the numerical results determines the true leading term of the expansion; after some work, we obtain

$$p_A(\rho) = -\left(\frac{3 - \sqrt{5}}{2\,\rho}\right)^{1/3} + \frac{35 + 12\,\sqrt{5}}{90} + O(\rho^{1/3}), \tag{68}$$

26

$$p_B(\rho) = - \left(\frac{\chi}{\rho} \right)^{1/3} + \alpha + O(\rho^{1/3}), \tag{69}$$

where

$$\chi = \frac{1}{15} \left\{ \left(\frac{2531 + 15\sqrt{31593}}{2} \right)^{1/3} - \left(\frac{-2531 + 15\sqrt{31593}}{2} \right)^{1/3} - 8 \right\} \tag{70}$$

is the real root of $5\chi^3 + 8\chi^2 + 8\chi - 1 = 0$. Numerically, $\chi \approx 0.11166155366$ and $\kappa = \chi^{1/3} \approx 0.48154242495$. The next-order calculation provides

$$p_B(\rho) = -\frac{\kappa}{\rho^{1/3}} + \underbrace{\frac{1}{50} \frac{70147451\,\kappa^2 + 22890531\,\kappa - 3689542}{1685743\,\kappa^2 + 778683\,\kappa - 121256}}_{\alpha} + O(\rho^{1/3}) \tag{71}$$

$$\approx -\frac{0.48154242495}{\rho^{1/3}} + 0.38969988720 + O(\rho^{1/3}). \tag{72}$$

Proceeding in a similar way for the critical points defining the rightmost structure in the complex plane, we find

$$p_D(\rho) = \left(\frac{3 - \sqrt{5}}{2\rho} \right)^{1/3} + \frac{19 + 4\sqrt{5}}{30} + O(\rho^{1/3}) \tag{73}$$

$$p_E(\rho) = \left(\frac{3 - \sqrt{5}}{2\rho} \right)^{1/3} + \frac{2}{15} \left(\frac{3 + \sqrt{5}}{2} \right)^{5/6} \sqrt{-75 + 35\sqrt{5}} \, \frac{1}{\rho^{1/6}}$$
$$+ \frac{11 + 4\sqrt{5}}{30} + O(\rho^{1/6}) \tag{74}$$

$$p_C(\rho) = e^{2 i \pi/3} \left(\frac{3 - \sqrt{5}}{2\rho} \right)^{1/3} + \frac{2}{15} \left(\frac{3 + \sqrt{5}}{2} \right)^{5/6} \sqrt{-75 + 35\sqrt{5}} \, \frac{e^{i\pi/3}}{\rho^{1/6}}$$
$$+ \frac{11 + 4\sqrt{5}}{30} + O(\rho^{1/6}) \tag{75}$$

Note that the expansion $p_C(\rho)$ is nothing but $p_E(\rho)$, with the transformation $\rho \longrightarrow e^{-2 i \pi} \rho$. It corresponds to the second complex root, by decreasing order of the real part.

5.3 Calculation for the fan

We expect here much easier calculations, since all the eigenvalues are extremely simple: 1, $p\,\rho$, and $p\,(1 - p)\,\rho$. They are, indeed, since the limiting curves can

be expressed analytically (see Table 3). It is worth noting that nonetheless, other structure transitions occur at $\rho = 1$ and $\rho = \frac{1}{2}$. Using the relevant recursion relation between successive reliability polynomials, we can again calculate $\mathrm{Rel}_2^{(\mathrm{fan})}(S_0 \to S_n)(p)$ rapidly using Mathematica.
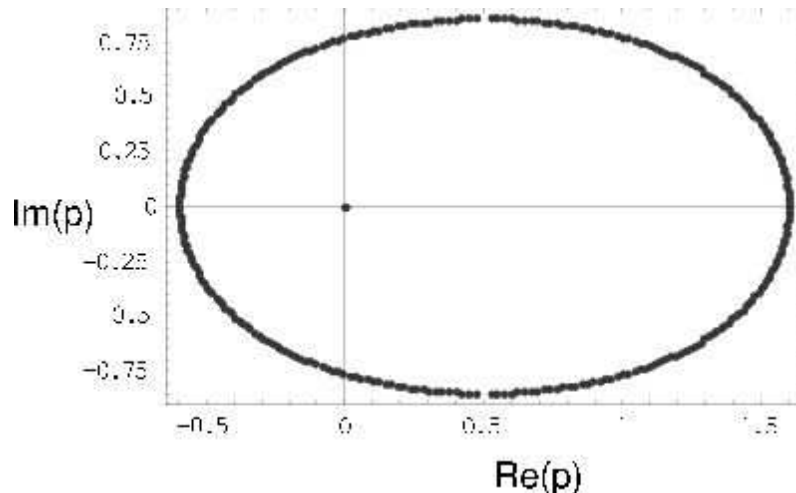

Fig. 11. Location of the roots of $\mathrm{Rel}_2^{(n=150)}(p) = 0$ for the fan, with $\rho = 1$.

For $\rho = 1$, we see on Figure 11 that the limiting curve is now essentially a closed one, even though the zeros appear to belong to two different sets which slowly join as n increases. The presence of n in a prefactor as in eqs. (59) and (60) does not affect the general definition of the limiting curves [8]. The limiting curve is very easy to obtain, since we have only two eigenvalues, 1 and $p(1-p)$. For them to have the same modulus implies $p(1-p) = -e^{i\theta}$, so that we get $p = \frac{1}{2}\left(1 \pm \sqrt{1 + 4\,e^{i\theta}}\right)$.

When ρ is strictly less than 1, the whole structure is altered, as witnessed by Figure 12 for $\rho = 0.9999$. Portions of circles centered at the origin and at the point $(1,0)$ are added. The reason for such additional structures comes from the existence of a new eigenvalue $p\rho$. They persist down to $\rho = \frac{1}{2}$. For $\rho < \frac{1}{2}$, the only eigenvalues that matter are actually 1 and $p(1-p)\rho$, so that the structure looks again very much like Fig. 11.

Their derivation being straightforward, we simply give the analytical expressions of the limiting curves in Table 3. When $\rho \to 0$, the structure expands with a scaling factor of $\rho^{-1/2}$, in contrast with the $\rho^{-1/3}$ obtained for the Brecht-Colbourn ladder.
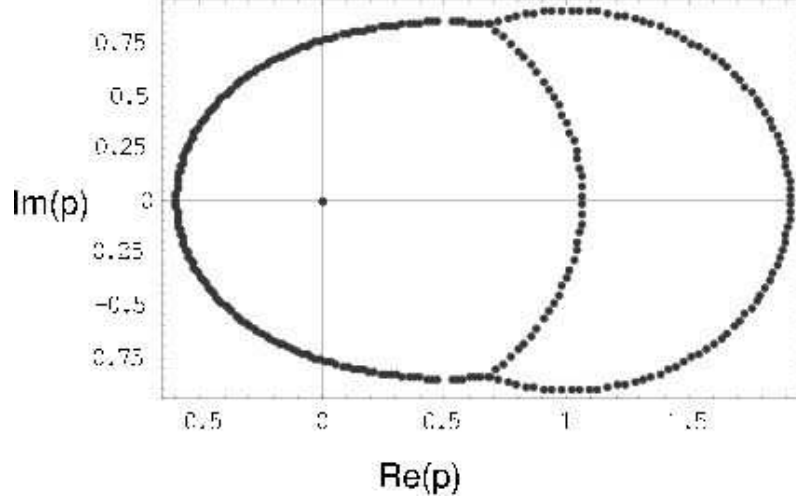
Fig. 12. Location of the roots of $\mathrm{Rel}_2^{(n=150)}(p) = 0$ for the fan, with $\rho = 0.9999$.

Table 3
Analytical expressions of the limiting aggregation curves for the location of p such that $\mathrm{Rel}_2^{(\mathrm{fan}),n\to\infty}(p,\rho) = 0$, as a function of ρ, the node reliability.

node reliability	parametrization	validity range
$\rho = 1$	$p = \frac{1}{2}\left(1 \pm \sqrt{1 + 4\,e^{i\theta}}\right)$	$-1 \leq \cos\theta \leq 1$
$\frac{1}{2} < \rho < 1$	$p = \frac{1}{2}\left(1 - \sqrt{1 + \frac{4}{\rho}\,e^{i\theta}}\right)$	$-1 \leq \cos\theta \leq 1$
	$p = \frac{1}{2}\left(1 + \sqrt{1 + \frac{4}{\rho}\,e^{i\theta}}\right)$	$-1 \leq \cos\theta \leq \frac{1}{2\rho}\left(\frac{1}{\rho^2} - 3\right)$
	$p = \frac{1}{\rho}\,e^{i\theta}$	$\cos\theta \geq \frac{1}{2\rho}$
	$p = 1 + e^{i\theta}$	$\cos\theta \geq \frac{1}{2\rho^2} - 1$
$0 < \rho \leq \frac{1}{2}$	$p = \frac{1}{2}\left(1 \pm \sqrt{1 + \frac{4}{\rho}\,e^{i\theta}}\right)$	$-1 \leq \cos\theta \leq 1$

6   All-terminal reliability

As mentioned in the introduction, the all-terminal reliability Rel_A is another useful measure of the network availability, giving the probability that all nodes are connected. In this case however, the node reliabilities are a mere overall factor, so that they may be considered equal to 1 for all practical purposes [26]. Chang and Shrock [19] gave the explicit expressions of Rel_A for various

29

recursive families of graphs, among which the Brecht-Colbourn ladder, with the same reliability p for all edges. The all-terminal reliability of the generalized fan has been calculated by Neufeld and Colbourn [49]. The results are of course identical, because at each step, two edges are added, while the common original graph is the triangle (the complete graph K_3). For the sake of completeness, we slightly generalize their result for edges with distinct reliabilities. Here again, the final, analytical expression can be written in a concise form using transfer matrices. In the context of graph theory, this can be viewed as the factorization of a particular value of the multi-variate Tutte polynomial, considered by Wu [74] and Sokal [64].
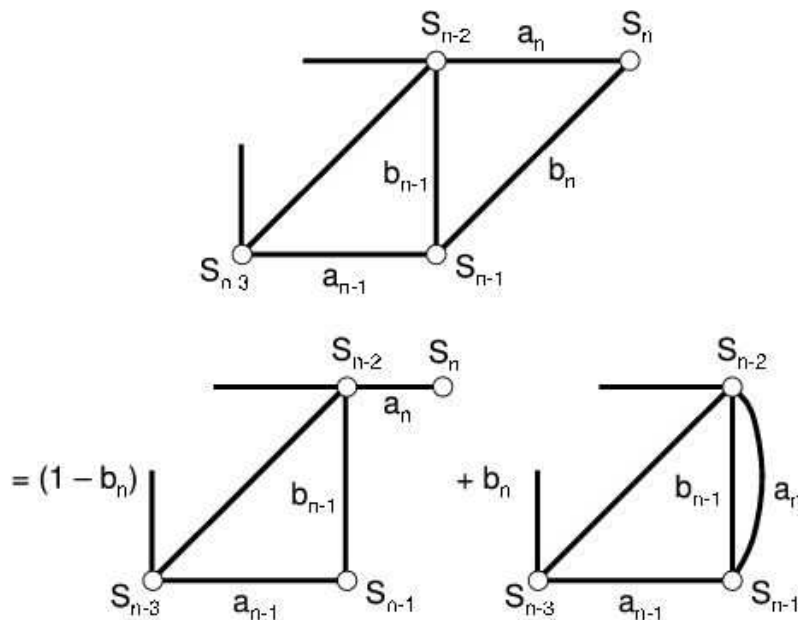

Fig. 13. Deletion-contraction simplification for the all-terminal reliability of the Brecht-Colbourn ladder.

We expect the all-terminal reliability $\mathcal{R}_n = \mathrm{Rel}_A(S_0 \leftrightarrow S_n)$ for the Brecht-Colbourn ladder to exhibit the same behavior as the two-terminal reliability, with a transfer matrix depending on a_n and b_n, and a generating function that is a rational fraction of p (the common reliability of links) and z. As mentioned above, this calculation should be somewhat easier because all nodes can be considered perfect without loss of generality [26]. We can then use the usual pivotal decomposition to establish a relationship between $\mathcal{R}_n$ and $\mathcal{R}_{n-1}$, as represented in Fig. 13. We find

$$\mathcal{R}_n = (1 - b_n)\, a_n\, \mathcal{R}_{n-1} + b_n\, \mathcal{R}_{n-1}(b_{n-1} \to b_{n-1} \,//\, a_n) \tag{76}$$

Using the ansatz $\mathcal{R}_n = \alpha_n\,(a_n + b_n) + \gamma_n\,a_n\,b_n$, we get

$$\begin{pmatrix} \alpha_n \\ \gamma_n \end{pmatrix} = \left(\begin{array}{c|c} a_{n-1} + b_{n-1} & a_{n-1}\,b_{n-1} \\ \hline 1 - a_{n-1} - 2\,b_{n-1} & a_{n-1}\,(1 - 2\,b_{n-1}) \end{array} \right) \begin{pmatrix} \alpha_{n-1} \\ \gamma_{n-1} \end{pmatrix} \tag{77}$$

with $a_1 = \gamma_1 = 0$ and $\alpha_1 = 1$. Calling $\widetilde{M}_{n-1}$ the transfer matrix of eq. (77), we obtain

$$\mathcal{R}_n = (1,0) \; \cdot \; \widetilde{M}_n \; \cdot \; \widetilde{M}_{n-1} \; \cdots \; \widetilde{M}_1 \; \cdot \; \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{78}$$

When all reliabilities are equal to p, the transfer matrix is simply

$$\widetilde{M'} = \left(\begin{array}{c|c} 2\,p & p^2 \\ \hline 1 - 3\,p & p\,(1 - 2\,p) \end{array} \right) \tag{79}$$

The eigenvalues of $\widetilde{M'}$ are $\zeta_{\pm} = \dfrac{p}{2}\,\left(3 - 2\,p \pm \sqrt{5 - 8\,p + 4\,p^2} \right)$, so that the generating function is

$$G_R(z) = \frac{p\,z^2}{1 - p\,(3 - 2\,p)\,z + p^2\,(1 - p)\,z^2} \tag{80}$$

and

$$\mathcal{R}_n = \frac{1}{\sqrt{5 - 8\,p + 4\,p^2}} \, \left(\zeta_+^{n-1} - \zeta_-^{n-1} \right) \qquad n \geq 2 \tag{81}$$

The last results were obtained in [19], while the same expression can be found in the paper by Neufeld and Colbourn [49].

7 Conclusion and Perspectives

We have given the exact solution of the two- and all-terminal reliabilities for the Brecht-Colbourn ladder and the generalized fan for arbitrary size and individual element reliability. While simple, these graphs correspond nonetheless to realistic network architectures, especially in telecommunication networks for IP transport. Node and edge failures are put on an equal footing, and the

31

simple formulae relying on transfer matrices may be directly implemented. We have also given the analytical solution of the two- and all-terminal reliabilities, when reliabilities are p and ρ for edges and nodes, along with their rational generating functions. The locations of the zeros of the two-terminal reliability polynomials differ for the two families of graphs considered in this paper, even though their all-terminal reliability is identical. They possess structures in the $\text{Re}(p) < 0$ region, and exhibit transitions for particular values of ρ.

Even though the delta-star transformation has also been successfully applied to the case of simple ladders [66], it may not be so easy to use in more complicated networks. However, it seems quite clear that a similar decomposition through transfer matrices of two-terminal reliabilities should occur for ladders of greater width too. In order to make such calculations useful for applications, imperfect nodes as well as imperfect edges must be considered. All is needed is a recursion relation between successive graphs, when one "elementary brick" is added. This implies a new expression for the deletion-contraction theorem, in which the linearity with respect to all individual edge or node reliabilities must be preserved, whereas — to our knowledge — edge reliabilities are often renormalized to account for the unreliability of the nodes they connect [3,70,71]. This new expression will be given elsewhere, along with an application to other recursive families of graphs such as the K_4 ladder and the K_3 cylinder [67].

What should we expect ? Basically, the same kind of behavior as detailed in the present work, with a factorization of the reliability in terms of transfer matrices, the dimension of which substantially increase to reflect the interplay of different edges/nodes in the overall reliability, and the underlying algebraic structure of the graph (one cannot escape the intrinsic complexity of the problem...).

There are obviously many directions in which this work may be further extended, for instance in the estimates of bounds. The first one is to use the present results, or their future extensions [67,69], as possible upper or lower bounds to more complex graphs. If a graph under consideration looks like — or made as such — a special instance of a recursive family of graphs, we expect the generating function to be a rational fraction again. The dimension of the corresponding transfer matrix may be probed by trying to find recursion relations between successive reliability polynomials. Of course, if the dimension of the corresponding transfer matrix is large — which is most likely to happen — the degree of the numerator and denominator of the fraction may be too large for a complete solution to be obtained easily. Even so, the knowledge that the *true* generating function is rational may be an indication that an *approximate* generating function might still be quite useful, because Padé approximants[4,50] are known for their devilish knack of getting *very* close to the exact function. This will be addressed elsewhere [69]. Our calculations

may also provide some information on some combinatorial issues, such as the enumeration of self-avoiding walks on lattices of restricted width.

Other quantities of interest may be deduced from the general expression of the network reliability, among them the sensitivity of a equipment [66], the influence of scheduled maintenance on the overall network availability, and the failure frequency [37,62], which can all be deduced from various partial derivatives of the two- and all-terminal reliabilities [68]. Since the reliability of each equipment appears in only one transfer matrix, the computation of all these network parameters is straightforward. One should also be aware that the reliability of each equipment is not known with absolute accuracy. The consequence of this uncertainty on the overall reliability has been addressed by Coit and collaborators in the case of series-parallel reducible networks [24,25]. Our transfer matrix factorization makes this topic another instance of a product of random matrices, definitely a vast field in mathematical physics [29].

Finally, the exact results found for classes of arbitrarily large networks may prove useful for testing different algorithms (Monte Carlo, genetic, OBDD, etc) in numerically exacting configurations, where edge and node unreliabilities must both be taken into account.

Acknowledgments

A Polynomials used in the determination of critical points for the zeros of $\mathrm{Rel}_2(p)$ (Brecht-Colbourn ladder)

A.1 $\mathcal{P}_1(p,\rho)$

$$
\begin{aligned}
\mathcal{P}_1(p,\rho) =\ & \rho - p\left(1 + 9\rho + 3\rho^2\right) + p^2\rho\left(41 + 37\rho - 11\rho^2\right) \\
& + p^3\rho\left(-39 - 195\rho + 99\rho^2 - 4\rho^3\right) \\
& + p^4\rho\left(16 + 147\rho - 411\rho^2 + 43\rho^3 + 11\rho^4\right) \\
& - p^5\rho^2\left(-54 - 2300\rho + 123\rho^2 + 67\rho^3 + 7\rho^4\right) \\
& - p^6\rho^2\left(126 + 4700\rho + 1865\rho^2 + 559\rho^3 - 59\rho^4 + 13\rho^5\right) \\
& + p^7\rho^2\left(41 + 4859\rho + 2324\rho^2 + 6400\rho^3 + 1135\rho^4 + 189\rho^5 + 14\rho^6\right) \\
& + p^8\rho^3\left(-2701 + 4149\rho - 11762\rho^2 - 8744\rho^3 - 2515\rho^4 - 311\rho^5 + 4\rho^6\right) \\
& - p^9\rho^3\left(-781 + 12546\rho - 6054\rho^2 - 14564\rho^3 - 11548\rho^4 - 3012\rho^5 - 124\rho^6 + 8\rho^7\right) \\
& - p^{10}\rho^3\left(96 - 13197\rho - 3996\rho^2 + 4727\rho^3 + 17901\rho^4 + 11943\rho^5 + 1464\rho^6 + 4\rho^7\right) \\
& + p^{11}\rho^4\left(-7185 - 3814\rho - 10626\rho^2 + 5695\rho^3 + 21337\rho^4 + 5871\rho^5 + 304\rho^6\right) \\
& - p^{12}\rho^4\left(-2040 + 3598\rho - 10478\rho^2 - 17573\rho^3 + 19235\rho^4 + 11669\rho^5 + 1219\rho^6\right) \\
& + p^{13}\rho^4\left(-240 + 6353\rho + 3091\rho^2 - 30176\rho^3 + 6625\rho^4 + 14726\rho^5 + 2128\rho^6\right) \\
& - p^{14}\rho^5\left(3613 + 12361\rho - 25852\rho^2 - 5221\rho^3 + 14188\rho^4 + 2147\rho^5\right) \\
& + p^{15}\rho^5\left(947 + 11148\rho - 14935\rho^2 - 8341\rho^3 + 9720\rho^4 + 2095\rho^5\right) \\
& - p^{16}\rho^5\left(96 + 5829\rho - 7507\rho^2 - 1909\rho^3 + 690\rho^4 + 3066\rho^5\right) \\
& + 2p^{17}\rho^6\left(993 - 2513\rho + 3568\rho^2 - 4056\rho^3 + 2046\rho^4\right) \\
& - p^{18}\rho^6\left(418 - 3991\rho + 10333\rho^2 - 10532\rho^3 + 3785\rho^4\right) \\
& + p^{19}\rho^6\left(41 - 2397\rho + 7184\rho^2 - 7257\rho^3 + 2430\rho^4\right) \\
& + 3p^{20}(1-\rho)\rho^7\left(299 - 684\rho + 367\rho^2\right) \\
& - p^{21}(1-\rho)\rho^7\left(185 - 519\rho + 322\rho^2\right) \\
& + p^{22}(1-\rho)\rho^7(1-2\rho)(16 - 15\rho) \\
& - 19p^{23}(1-\rho)^2\rho^8 + 8p^{24}(1-\rho)^2\rho^8 - p^{25}(1-\rho)^2\rho^8
\end{aligned}
\tag{A.1}
$$

A.2 $\mathcal{P}_2(p,\rho)$

$$
\mathcal{P}_2(p,\rho) =
$$

$$
9 + 2p\left(-93 + 37\rho\right) + p^2\left(1375 - 1139\rho + 235\rho^2\right)
$$

$$+p^3 \left(-4548 + 6619\,\rho - 2306\,\rho^2 + 380\,\rho^3\right)$$

$$+p^4 \left(7551 - 22762\,\rho + 7011\,\rho^2 - 2415\,\rho^3 + 335\,\rho^4\right)$$

$$+p^5 \left(-6210 + 53479\,\rho + 2160\,\rho^2 + 7102\,\rho^3 - 1462\,\rho^4 + 154\,\rho^5\right)$$

$$+p^6 \left(2025 - 85209\,\rho - 79460\,\rho^2 - 18121\,\rho^3 - 11141\,\rho^4 + 223\,\rho^5 + 29\,\rho^6\right)$$

$$+p^7\,\rho \left(84879 + 277027\,\rho + 106511\,\rho^2 + 202610\,\rho^3 - 36581\,\rho^4 + 1170\,\rho^5\right)$$

$$+p^8\,\rho \left(-46170 - 521894\,\rho - 587852\,\rho^2 - 1536749\,\rho^3 + 263741\,\rho^4 - 39391\,\rho^5 + 547\,\rho^6\right)$$

$$-p^9\,\rho \left(-10125 - 611502\,\rho - 1935829\,\rho^2 - 7279427\,\rho^3 + 440373\,\rho^4 \right.$$
$$\left. -331984\,\rho^5 + 16780\,\rho^6\right)$$

$$+p^{10}\,\rho^2 \left(-455796 - 3964687\,\rho - 23412205\,\rho^2 - 3663927\,\rho^3 - 1452867\,\rho^4 \right.$$
$$\left. +375501\,\rho^5 + 2506\,\rho^6\right)$$

$$+p^{11}\,\rho^2 \left(204255 + 5324974\,\rho + 53128592\,\rho^2 + 28420289\,\rho^3 + 5054943\,\rho^4 \right.$$
$$\left. -4185113\,\rho^5 + 12850\,\rho^6\right)$$

$$+p^{12}\,\rho^2 \left(-42525 - 4776213\,\rho - 87000246\,\rho^2 - 104194583\,\rho^3 \right.$$
$$\left. -21210055\,\rho^4 + 27622695\,\rho^5 - 279589\,\rho^6 + 23497\,\rho^7\right)$$

$$+p^{13}\,\rho^3 \left(2798298 + 103839446\,\rho + 248730949\,\rho^2 + 91081790\,\rho^3 \right.$$
$$\left. -119321953\,\rho^4 - 214483\,\rho^5 + 2402\,\rho^6\right)$$

$$+p^{14}\,\rho^3 \left(-987660 - 90012828\,\rho - 421343630\,\rho^2 - 300983008\,\rho^3 \right.$$
$$\left. +354546288\,\rho^4 + 19055623\,\rho^5 - 2902607\,\rho^6 + 156356\,\rho^7\right)$$

$$-p^{15}\,\rho^3 \left(-164025 - 55522953\,\rho - 523811627\,\rho^2 - 718737692\,\rho^3 \right.$$
$$\left. +737559653\,\rho^4 + 137144016\,\rho^5 - 24588235\,\rho^6 + 1508228\,\rho^7\right)$$

$$+p^{16}\,\rho^4 \left(-23264478 - 483102731\,\rho - 1256125498\,\rho^2 + 1050744677\,\rho^3 \right.$$
$$\left. +534128347\,\rho^4 - 99647028\,\rho^5 + 3934654\,\rho^6 + 455372\,\rho^7\right)$$

$$-p^{17}\,\rho^4 \left(-5993055 - 328891636\,\rho - 1637004249\,\rho^2 + 897069980\,\rho^3 \right.$$
$$\left. +1365985483\,\rho^4 - 226306853\,\rho^5 - 13885678\,\rho^6 + 5675792\,\rho^7\right)$$

$$+p^{18}\,\rho^4 \left(-729000 - 161715291\,\rho - 1608420692\,\rho^2 + 82688597\,\rho^3 + 2449767599\,\rho^4 \right.$$
$$\left. -232309325\,\rho^5 - 139531559\,\rho^6 + 32366472\,\rho^7 + 274104\,\rho^8\right)$$

$$-p^{19}\,\rho^5 \left(-54779328 - 1192868328\,\rho - 1036350248\,\rho^2 + 3126514952\,\rho^3 \right.$$
$$\left. +252124085\,\rho^4 - 530576442\,\rho^5 + 110376920\,\rho^6 + 3523400\,\rho^7\right)$$

$$+p^{20}\,\rho^5 \left(-11563290 - 661113157\,\rho - 1778966082\,\rho^2 + 2728489572\,\rho^3 \right.$$
$$\left. +1464568704\,\rho^4 - 1261931324\,\rho^5 + 243789711\,\rho^6 + 21533484\,\rho^7\right)$$

$$-p^{21}\,\rho^5 \left(-1166400 - 267060831\,\rho - 1769360761\,\rho^2 + 1310882671\,\rho^3 \right.$$
$$\left. +3038379428\,\rho^4 - 2084661610\,\rho^5 + 338855326\,\rho^6 + 82890944\,\rho^7\right)$$

$$+p^{22}\,\rho^6 \left(-74824812 - 1221905422\,\rho - 276377214\,\rho^2 + 4066473373\,\rho^3 \right.$$
$$\left. -2443698651\,\rho^4 + 202709670\,\rho^5 + 224730742\,\rho^6\right)$$

$$-p^{23}\,\rho^6 \left(-13126860 - 609332834\,\rho - 1166508009\,\rho^2 + 3871377708\,\rho^3 \right.$$
$$\left. -1932786432\,\rho^4 - 295096153\,\rho^5 + 454945390\,\rho^6\right)$$

$$+p^{24}\,\rho^6 \left(-1103625 - 218082828\,\rho - 1184923715\,\rho^2 + 2650067734\,\rho^3 \right.$$

$$-752045795\,\rho^4 - 1022894704\,\rho^5 + 711726527\,\rho^6\big)$$

$$-p^{25}\,\rho^7\,\big(-53743284 - 754572613\,\rho + 1226487190\,\rho^2 + 416790514\,\rho^3$$
$$-1614449937\,\rho^4 + 878716880\,\rho^5\big)$$

$$+p^{26}\,\rho^7\,\big(-8258625 - 336506619\,\rho + 274671788\,\rho^2 + 1010093115\,\rho^3$$
$$-1758220080\,\rho^4 + 866894232\,\rho^5\big)$$

$$-p^{27}\,\rho^7\,\big(-605475 - 106456428\,\rho - 91817943\,\rho^2 + 978393477\,\rho^3$$
$$-1446294368\,\rho^4 + 687560517\,\rho^5\big)$$

$$+p^{28}\,\rho^8\,\big(-23082921 - 120749456\,\rho + 640575958\,\rho^2 - 928058750\,\rho^3 + 438708098\,\rho^4\big)$$

$$-p^{29}\,\rho^8\,\big(-3116070 - 61389864\,\rho + 311935039\,\rho^2 - 469461042\,\rho^3 + 224128938\,\rho^4\big)$$

$$+p^{30}\,\rho^8\,\big(-200475 - 19494900\,\rho + 116248678\,\rho^2 - 186770845\,\rho^3 + 90668832\,\rho^4\big)$$

$$-p^{31}\,\rho^9\,\big(-4042170 + 33260499\,\rho - 57626524\,\rho^2 + 28477028\,\rho^3\big)$$

$$+p^{32}\,\rho^9\,\big(-510300 + 7194033\,\rho - 13394778\,\rho^2 + 6717655\,\rho^3\big)$$

$$-3\,p^{33}\,\rho^9\,\big(-10125 + 376209\,\rho - 740598\,\rho^2 + 374614\,\rho^3\big)$$

$$+540\,p^{34}\,(1-\rho)\,\rho^{10}\,(216 - 221\,\rho) - 6075\,p^{35}\,(1-\rho)^2\,\rho^{10}. \tag{A.2}$$

A.3 $\mathcal{P}_3(p,\rho,T)$

$$\mathcal{P}_3(p,\rho,T) =$$

$$2 + 2\,T + p\,\big(-6 - 10\,T - 4\,T^2\big)$$
$$+p\,\rho\,\Big[5 + 12\,T - 8\,T^3 + p\,\big(-29 - 52\,T + 24\,T^2 + 64\,T^3 + 16\,T^4\big)$$
$$+p^2\,\big(78 + 138\,T - 52\,T^2 - 176\,T^3 - 64\,T^4\big)$$
$$+p^3\,\big(-36 - 60\,T + 48\,T^2 + 120\,T^3 + 48\,T^4\big)\Big]$$
$$+p^2\,\rho^2\,\Big[4\,T + 8\,T^2 - 8\,T^3 - 16\,T^4 + p\,\big(-8 + 8\,T - 24\,T^2 + 48\,T^4\big)$$
$$+p^2\,\big(56 - 22\,T + 44\,T^2 + 192\,T^3 + 128\,T^4 + 64\,T^5\big)$$
$$+p^3\,\big(-262 - 336\,T + 40\,T^3 - 256\,T^4 - 288\,T^5 - 64\,T^6\big)$$
$$+p^4\,\big(248 + 384\,T - 120\,T^2 - 384\,T^3 + 128\,T^4 + 384\,T^5 + 128\,T^6\big)$$
$$+p^5\,\big(-66 - 110\,T + 52\,T^2 + 160\,T^3 - 32\,T^4 - 160\,T^5 - 64\,T^6\big)\Big]$$
$$+p^3\,\rho^3\,\Big[-10 - 40\,T - 40\,T^2 + 8\,T^3 + 16\,T^4$$
$$+p\,\big(58 + 164\,T + 264\,T^2 + 24\,T^3 - 256\,T^4 - 128\,T^5\big)$$
$$+p^2\,\big(-206 - 378\,T - 800\,T^2 - 656\,T^3 + 416\,T^4 + 640\,T^5 + 192\,T^6\big)$$
$$+p^3\,\big(448 + 890\,T + 1296\,T^2 + 1080\,T^3 - 256\,T^4 - 864\,T^5 - 384\,T^6\big)$$
$$+p^4\,\big(-237 - 668\,T - 1188\,T^2 - 920\,T^3 - 32\,T^4 + 320\,T^5 + 192\,T^6\big)$$

$$+p^5 \left(-151 - 70\,T + 760\,T^2 + 904\,T^3 + 256\,T^4 + 32\,T^5\right)$$
$$+p^6 \left(162 + 230\,T - 300\,T^2 - 560\,T^3 - 192\,T^4\right)$$
$$\left.+p^7 \left(-36 - 60\,T + 48\,T^2 + 120\,T^3 + 48\,T^4\right)\right]$$

$$+p^4\,\rho^4 \left[-10 - 54\,T - 104\,T^2 - 56\,T^3 + 80\,T^4 + 64\,T^5\right.$$

$$+p \left(54 + 274\,T + 492\,T^2 + 400\,T^3 - 144\,T^4 - 416\,T^5 - 192\,T^6\right)$$
$$+p^2 \left(-12 - 578\,T - 1244\,T^2 - 904\,T^3 + 48\,T^4 + 576\,T^5 + 384\,T^6\right)$$
$$+p^3 \left(-206 + 456\,T + 2000\,T^2 + 1688\,T^3 + 304\,T^4 - 160\,T^5 - 192\,T^6\right)$$
$$+p^4 \left(156 - 356\,T - 2060\,T^2 - 2120\,T^3 - 592\,T^4 - 64\,T^5\right)$$
$$+p^5 \left(92 + 462\,T + 1228\,T^2 + 1272\,T^3 + 400\,T^4\right)$$
$$+p^6 \left(-88 - 270\,T - 372\,T^2 - 288\,T^3 - 96\,T^4\right)$$
$$+p^7 \left(-22 - 2\,T + 28\,T^2 + 8\,T^3\right)$$
$$+p^8 \left(30 + 46\,T + 16\,T^2\right)$$
$$\left.+p^9 \left(-6 - 10\,T - 4\,T^2\right)\right]$$

$$+p^5\,\rho^5 \left[-3 - 20\,T - 56\,T^2 - 64\,T^3 - 16\,T^4 + 64\,T^5 + 64\,T^6\right.$$

$$+p \left(11 + 160\,T + 432\,T^2 + 376\,T^3 + 80\,T^4 - 96\,T^5 - 128\,T^6\right)$$
$$+p^2 \left(-84 - 488\,T - 1116\,T^2 - 984\,T^3 - 240\,T^4 + 64\,T^6\right)$$
$$+p^3 \left(240 + 850\,T + 1416\,T^2 + 1224\,T^3 + 336\,T^4 + 32\,T^5\right)$$
$$+p^4 \left(-252 - 790\,T - 960\,T^2 - 712\,T^3 - 208\,T^4\right)$$
$$+p^5 \left(74 + 318\,T + 328\,T^2 + 168\,T^3 + 48\,T^4\right)$$
$$+p^6 \left(43 + 14\,T - 28\,T^2 - 8\,T^3\right)$$
$$+p^7 \left(-33 - 48\,T - 16\,T^2\right)$$
$$\left.+p^8 \left(6 + 10\,T + 4\,T^2\right)\right] \tag{A.3}$$

When $T = -1$, we can further simplify the above expression

$$\mathcal{P}_3(p, \rho, T = -1) = (1 - p)\, p\, \rho \left(1 + p\,(1 - p)\,\rho\right)^2 \left(1 - \rho + \rho\,(1 - p)^3\right)^2 \tag{A.4}$$

There is no such factorization in the case $T = 1$:

$$\mathcal{P}_3(p, \rho, T = 1) =$$

$$4 - 20\,p + p\,\rho \left(9 + 23\,p - 76\,p^2 + 120\,p^3\right)$$
$$-2\,p^2\,\rho^2 \left(6 - 12\,p - 231\,p^2 + 583\,p^3 - 384\,p^4 + 110\,p^5\right)$$
$$+p^3\,\rho^3 \left(-66 + 126\,p - 792\,p^2 + 2210\,p^3 - 2533\,p^4 + 1731\,p^5 - 660\,p^6 + 120\,p^7\right)$$
$$-2\,p^4\,\rho^4 \left(40 - 234\,p + 865\,p^2 - 1945\,p^3 + 2518\,p^4 - 1727\,p^5 + 557\,p^6 - 6\,p^7 - 46\,p^8 + 10\,p^9\right)$$

$$+p^5 \rho^5 \left(-31 + 835\,p - 2848\,p^2 + 4098\,p^3 - 2922\,p^4 + 936\,p^5 + 21\,p^6 - 97\,p^7 + 20\,p^8\right) \quad \text{(A.5)}$$

References

[1] H. M. AboElFotoh and C. J. Colbourn, Computing 2-terminal reliability for radio-broadcast networks, IEEE Trans. Reliability 38 (5) (1989), 538–555.

[2] J. A. Abraham, An improved algorithm for network reliability, IEEE Trans. Reliability 28 (1) (1979), 58–61.

[3] K. K. Aggarwal, J. S. Gupta, and K. B. Misra, A simple method for reliability evaluation of a communication system, IEEE Trans. Communications 23 (1975), 563–566.

[4] G. A. Baker, Jr. and P. Graves-Morris, Padé approximants, 2nd edition, Cambridge University Press, Cambridge, 1996.

[5] A. O. Balan and L. Traldi, Preprocessing minpaths for sum of disjoint products, IEEE Trans. Reliability 52 (3) (2003), 289–295.

[6] M. O. Ball, C. J. Colbourn, and J. Scott Provan, "Network reliability," Handbooks in operations research and management science, volume 7 : Network Models, M. O. Ball, T. L. Magnanti, C. L. Monma, and G. L. Nemhauser (Editors), Elsevier, Amsterdam, 1995, pp. 673–762 (with more than 400 references).

[7] R. E. Barlow, F. Proschan, and L. C. Hunter, Mathematical theory of reliability, Wiley, New York, 1965.

[8] S. Beraha, J. Kahane, and N. J. Weiss, Limits of zeros of recursively defined families of polynomials, Studies in Foundations and Combinatorics, Advances in Mathematics Supplementary Studies, vol. 1, G.-C. Rota, Editor, pp. 213–232, Academic Press, New York, 1978.

[9] N. L. Biggs and G. H. J. Meredith, Approximations for chromatic polynomials, J. Combin. Theory B 20 (1976), 5–19.

[10] N. Biggs, Algebraic graph theory, 2nd edition, Cambridge University Press, Cambridge, 1993.

[11] N. L. Biggs, Matrix method for chromatic polynomials, J. Combin. Theory B 82 (2001), 19–29.

[12] F. Boesch, A. Satyanarayana, and C. Suffel, "On residual connectedness network reliability", Reliability of Computer and Communication Networks (DIMACS 5), New Brunswick, New Jersey, USA, pp. 51–59, F. S. Roberts, F. Hwang, and C. L. Monma, Eds, American Mathematical Society, 1991.

[13] T. B. Brecht, Lower bounds for two-terminal network reliability, Master of Mathematics thesis, University of Waterloo, Ontario, 1985.

[14] T. B. Brecht and C. J. Colbourn, Improving reliability bounds in computer networks, Networks 16 (1986), 369–380.

[15] T. B. Brecht and C. J. Colbourn, Lower bounds on two-terminal network reliability, Discrete Applied Mathematics, vol. 21, pp. 185–198, 1988.

[16] J. I. Brown and C. J. Colbourn, Roots of the reliability polynomial, SIAM J. Disc. Math. 5 (4) (1992), 571–585.

[17] J. I. Brown and C. J. Colbourn, Non-Stanley bounds for network reliability, J. Algebraic Combinatorics 5 (1996), 13–36.

[18] D. Bulka and J. B. Dugan, Network $s-t$ reliability bounds using a 2-dimensional reliability polynomial, IEEE Trans. Reliability 43 (1) (1994), 39–45.

[19] S.-C. Chang and R. Shrock, Reliability polynomials and their asymptotic limits for families of graphs, J. Statist. Phys. 112 (2003), 1019–1077; arXiv:cond-mat/0208538.

[20] M. K. Chari, T. A. Feo, and J. Scott Provan, The delta-wye approximation procedure for two-terminal reliability, UNC/OR TR92-18; Operation Research 44 (1996), 745–755, and references therein.

[21] M. K. Chari and C. J. Colbourn, Reliability polynomials : a survey, J. Combin. Inform. System Sci. 22 (1997), 177–193; August 12, 1998 Web version.

[22] Yinong Chen and Zhongshi He, Bounds on the reliability of distributed systems with unreliable nodes & links, IEEE Trans. Reliability 53 (2) (2004), 205–215.

[23] D. W. Coit and A. E. Smith, Reliability optimization of series-parallel systems using a genetic algorithm, IEEE Trans. Reliability, 45 (2) (1996), 254–260, 266.

[24] D. W. Coit, System-reliability confidence-intervals for complex-systems with estimated component-reliability, IEEE Trans. Reliability 46 (4) (1997) 487–493.

[25] D. W. Coit, T. Jin, and N. Wattanapongsakorn, System optimization with component reliability estimation uncertainty : a multi-criteria approach, IEEE Trans. Reliability, 53 (3) (2004), 369–380.

[26] C. J. Colbourn, The combinatorics of network reliability, Oxford University Press, Oxford, 1987.

[27] C. J. Colbourn and E. I. Litvak, "Bounding Network parameters by approximating graphs", Reliability of Computer and Communication Networks (DIMACS 5), New Brunswick, New Jersey, USA, pp. 91–104, F. S. Roberts, F. Hwang, and C. L. Monma, Eds, American Mathematical Society, 1991.

[28] C. J. Colbourn, Some open problems on reliability polynomials, DIMACS technical report 93-28, April 1993; Congressus Numerantium 93 (1993), 187–202.

[29] A. Crisanti, G. Paladin, and A. Vulpiani, Products of random matrices in statistical physics, Springer, Berlin, 1993.

[30] W. P. Dotson and J. O. Gobien, A new analysis technique for probabilistic graphs, IEEE Trans. Circuits and Systems 26 (10) (1979), 855–865.

[31] T. Egeland and A. B. Huseby, On dependence and reliability computations, Networks 21 (1991), 521–545.

[32] T. Evans and D. Smith, Optimally reliable graphs for both edge and vertex failures, Networks 16 (1986), 199–204.

[33] G. S. Fishman, A comparison of four Monte Carlo methods for estimating the probability of $s - t$ connectedness, IEEE Trans. Reliability 35 (2) (1986), 145–155.

[34] J. P. Gadani, System effectiveness evaluation using star and delta transformations, IEEE Trans. Reliability 30 (1) (1981), 43–47.

[35] J. Graver and M. Sobel, You may rely on the reliability polynomial for much more than you might think, Communications in Statistics: Theory and Methods 34 (6) (2005), 1411–1422.

[36] E. Hänsler, G. K. McAuliffe, and R. S. Wilkov, "Exact calculation of Computer Network Reliability," Networks 4 (1974), 95–112.

[37] M. Hayashi, System failure-frequency analysis using a differential operator, IEEE Trans. Reliability 40 (5) (1991), 601–609,614.

[38] K. D. Heidtmann, Smaller sums of disjoint products by subproduct inversion, IEEE Trans. Reliability, 38 (3) (1989), 305–311.

[39] T. Jin and D. W. Coit, Variance of system-reliability estimates with arbitrarily repeated components, IEEE Trans. Reliability 50 (4) (2001) 409–413.

[40] D. R. Karger, A randomized fully polynomial time approximation scheme for the all terminal reliability problem, SIAM Review vol. 43 (3) (2001), 499–522.

[41] W.-J. Ke and S.-D. Wang, Reliability evaluation for distributed computing networks with imperfect nodes, IEEE Trans. Reliability 46 (3) (1993), 342–349.

[42] R. Kevin Wood, A factoring algorithm using polygon-to-chain reductions for computing K-terminal network reliability, Networks 15 (1985), 173–190.

[43] S. Kuo, S. Lu, and F. Yeh, Determining terminal pair reliability based on edge expansion diagrams using OBDD, IEEE Trans. Reliability 48 (3) (1999), 234–246.

[44] S. Liu, K.-H. Cheng, and X. Liu, Network reliability with node failures, Networks 35 (2) (2000), 109–117.

[45] Mathematica, Wolfram Research, version 5.

[46] E. F. Moore and C. E. Shannon, Reliable circuits using less reliable relays, Journal Franklin Institute, 262 (September 1956), 191–208; 262 (October 1956), 281–297.

[47] L. D. Nel and C. J. Colbourn, Combining Monte Carlo estimates and bounds for network reliability, Networks 20 (1990), 277–298.

[48] V. A. Netes and B. P. Filin, Consideration of node failures in network-reliability calculation, IEEE Trans. Reliability, 45 (1) (1996), 127–128.

[49] E. M. Neufeld and C. J. Colbourn, The most reliable series-parallel networks, Networks 15 (1985), 27–32.

[50] W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling, Numerical recipes in C : the art of scientific computing, 2nd edition, chapter 5.12, Cambridge University Press, Cambridge, 1992.

[51] J. Oxley and D. Welsh, Chromatic, flow and reliability polynomials: the complexity of their coefficients, Combinatorics, Probability and Computing 11 (2002), 403–426.

[52] A. Prékopa, E. Boros, and K.-W. Lih, The use of binomial moments for bounding network reliability", Reliability of Computer and Communication Networks (DIMACS 5), New Brunswick, New Jersey, USA, pp. 197–212, F. S. Roberts, F. Hwang, and C. L. Monma, Eds, American Mathematical Society, 1991.

[53] S. Rai, M. Veeraraghavan, and K. S. Trivedi, A survey of efficient reliability computation using disjoint products approach, Networks 25 (1995), 147–163, and references therein.

[54] A. Rauzy, A new methodology to handle Boolean models with loops, IEEE Trans. Reliability 52 (1) (2003) 96–105.

[55] A. Rosenthal and D. Frisque, Transformations for simplifying network reliability calculations, Networks 7 (1977), 97–111; errata, Networks 7 (1977), 382.

[56] G. F. Royle and A. D. Sokal, The Brown-Colbourn conjecture on zeros of reliability polynomials is false, J. Combin. Theory B 91 (2) (2004), 345–360; arXiv:math.CO/0301199.

[57] J. Salas and A. D. Sokal, Transfer matrices and partition-function zeros for antiferromagnetic Potts models. I. General theory and square-lattice chromatic polynomial, J. Statist. Phys. 104 (2001), 609–699; cond-mat/0004330.

[58] J. Scott Provan, Bounds on the reliability of networks, IEEE Trans. Reliability R-35 (3) (1986), 260–268.

[59] D. R. Shier, Network reliability and algebraic structures, Clarendon Press, Oxford, 1991.

[60] M. L. Shooman, Probabilistic reliability: an engineering approach, McGraw-Hill, New York, 1968.

[61] R. Shrock, Exact Potts model partition functions on ladder graphs, Physica A283 (2000), 388–446 ; arXiv:cond-mat/0001389.

[62] C. Singh and R. Billinton, System reliability modelling and evaluation, Hutchinson, London, 1977.

[63] S. Soh and S. Rai, Experimental results on preprocessing of path/cut terms in sum of disjoint products technique, IEEE Trans. Reliability 42 (1) (1993), 24–33.

[64] A. D. Sokal, The multivariate Tutte polynomial (alias Potts model) for graphs and matroids, Surveys in Combinatorics 2005, London Mathematical Society Lecture Note Series 327, ed. Bridget S. Webb, pp. 173–226, (Cambridge University Press, Cambridge, 2005); arXiv:math.CO/0503607.

[65] R. P. Stanley, Enumerative combinatorics, vol. 1, chapter 4, Cambridge University Press, Cambridge, 1997.

[66] C. Tanguy, Exact solutions for the two- and all-terminal reliabilities of a simple ladder network, submitted to Networks.

[67] C. Tanguy, Transfer matrix method for two- and all-terminal reliabilities in ladder networks, to be submitted.

[68] C. Tanguy, Dispersive effects in network reliability, to be submitted.

[69] C. Tanguy, in preparation.

[70] O. R. Theologou and J. G. Carlier, Factoring & reductions for networks with imperfect vertices, IEEE Trans. Reliability 40 (2) (1991), 210–217.

[71] D. Torrieri, Calculation of node-pair reliability in large networks with unreliable nodes, IEEE Trans. Reliability 43 (3) (1994), 375–377,382.

[72] S.-D. Wang and C.-H. Sun, Transformations of star-delta and delta-star reliability networks, IEEE Trans. Reliability 45 (1) (1996), 120–126.

[73] D. J. A. Welsh and C. Merino, The Potts model and the Tutte polynomial, J. Math. Phys. 41 (1) (2000), 1127–1152.

[74] F. Y. Wu, "Graph theory in statistical physics," Studies in Foundations and Combinatorics, Advances in Mathematics Supplementary Studies, vol. 1, G.-C. Rota, Editor, pp. 213–232, Academic Press, New York, 1978.

[75] F. M. Yeh, S. K. Lu, and S. Y. Kuo, OBDD-based evaluation of k-terminal network reliability, IEEE Trans. Reliability 51 (4) (2002) 443–451.

[76] Fu-Min Yeh, Hung-Yau Lin, and Sy-Yen Kuo, Analyzing network reliability with imperfect nodes using OBDD, Proceedings of the 2002 Pacific Rim International Symposium on Dependable Computing (PRDC'02), pp. 89–96.

[77] Y. B. Yoo and N. Deo, A comparison of algorithms for terminal-pair reliability, IEEE Trans. Reliability 37 (2) (1988), 210–215.